

# A group theoretical approach to causal structures and positive energy on spacetimes

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## Abstract

This article presents a precise description of the interplay between the symmetries of a quantum or classical theory with spacetime interpretation, and some of its physical properties relating to causality, horizons and positive energy. Our major result is that the existence of static metrics on spacetimes and that of positive energy representations of symmetry groups, are equivalent to the existence of particular Adjoint-invariant convex cones in the symmetry algebras. This can be used to study backgrounds of supergravity and string theories through their symmetry groups. Our formalism is based on Segal's approach to infinitesimal causal structures on manifolds. The Adjoint action in the symmetry group is shown to correspond to changes of inertial frames in the spacetime, whereas Adjoint-invariance encodes invariance under changes of observers. This allows us to give a group theoretical description of the horizon structure of spacetimes, and also to lift causal structures to the Hilbert spaces of quantum theories. Among other results, by setting up the Dirac procedure for the complexified universal algebra, we classify the physically inequivalent observables of quantum theories. We illustrate this by finding the different Hamiltonians for stationary observers in  $AdS_2$ .

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## 1 Introduction

The considerable variety of backgrounds allowed in supergravity or string theories, including spaces with closed timelike curves [1, 2], or with various types of singularities [2–6], suggests it would be useful to analyse physically relevant properties of spacetimes more systematically and methodically, in order to obtain a categorisation of spacetimes as backgrounds hosting physically different theories. This is further motivated by the difficulty to reconcile current theories, or even elementary notions of quantum field theory, with a positive cosmological constant. Part of this incompatibility results from the fact that supersymmetric bosonic solutions of supergravity admit a causal Killing field [7], whereas spacetimes with a positive cosmological constant seem not to. The difference of impetus between the  $AdS/CFT$  and  $dS/CFT$  conjectures also illustrates a fundamental discrepancy between negatively and positively curved spaces. Symmetries and supersymmetries of spacetimes play a decisive rôle in determining essential physical properties such as conservation and boundedness of the energy [8], stability to perturbations, the existence of horizons or Ergo-regions, and the existence of closed timelike curves [9–11]. In fact we shall see that many “physical properties” of a theory, whether classical or quantum, such as the classification of inequivalent Hamiltonians, or the possibility to define a positive energy, are encoded directly in its

group of symmetries: its structure thus appears to be as fundamental as the defining equations of the theory.

Felix Klein, when he launched the “Erlanger Programm” [12] in 1872, believed that geometry itself should consist of the study of properties of a space that are invariant under a group of transformations. The causal structure of a spacetime –by which we mean the transitive relation between points which are connected by timelike or null curves–, encodes some physical properties of theories built on this spacetime: for example, non-staticity of spacetime metrics introduces horizons which can radiate or more simply laws of physics which appear to be time-varying. We shall lift the concept of causal structures to the Hilbert spaces of quantum theories with spacetime interpretation. Klein’s motto suggests the causal structures should highly depend on their groups of symmetries. A well-known example is provided by the comparison between Anti-de-Sitter and de-Sitter spacetimes. Considered as the homogeneous manifolds  $AdS_{n+1} = SO(2, n)/SO(1, n)$  and  $dS_{n+1} = SO(1, n+1)/SO(1, n)$ , they have important structural differences. Philips and Wigner [13] showed that all the conjugacy classes of the de Sitter groups  $O(1, n+1)$  are ambivalent<sup>1</sup> if  $n \equiv 0$  or  $3$  modulo  $4$ , while this is not the case in the Anti-de-Sitter group  $O(2, n)$ . We know that  $AdS_{n+1}$  admits a notion of positive energy, whereas  $dS_{n+1}$  does not. By endowing the symmetry group of a theory with a causal structure induced from that of spacetime, we will explain precisely the interplay between the Adjoint action in the group and spacetime physics, and in particular explain generalisations of the previous comment on de-Sitter space. It will turn out particular cones in the Lie algebra of the symmetries play a major role, both in quantum and classical theories. The group theoretical formalism we develop, though more stringent and hence interesting for homogeneous spacetimes, can be used in any spacetime theory with at least one symmetry. A typical set of differential equations will often have at least one symmetry, and hence our results have a wide application.

Segal was the first to analyse causal structures on spaces in a Klein fashion: following Vinberg’s work [14] on convex cones and obviously motivated by the future cones in General Relativity, he defined [15] an infinitesimal causal structure on a manifold to be a smooth assignment of a convex cone at each point. He analysed which symmetry groups could act on such infinitesimal structures and yield (global) causal structures which have no closed causal curves. Precisely, he focused on the Lie groups admitting infinitesimal causal structures which are defined by an Adjoint-invariant cone in their Lie algebra, but did not really consider homogeneous spaces of such Lie groups. Moreover, Segal did not justify his assumptions by physical motivations, and it seems that his followers in this subject, apart from maybe Paneitz [16, 17], lead this topic into the mathematical realm. As a consequence considerable mathematical progress has been done since the work of Segal, particularly on the causal structures of symmetric spaces. A recent account of the field is given in [18], and other related topics in [19]. The theory has deep connections with other areas of mathematics such as semi-groups, hermitian symmetric domains, unitary representations, many of which commonly have applications in modern theoretical physics. However, the assumptions of Segal, and hence most of his results on causal structures, have not been given a physical sense. The first aim of this article is to do so.

Starting from general considerations about the symmetries of a theory with a space-

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<sup>1</sup>That is, every element of the group is in the same class as its inverse, or every vector in the Lie algebra can be transformed to its opposite by the Adjoint action of a group element.

time interpretation, whether of particles, fields, or extended objects, we show that these must induce spacetime diffeomorphisms which preserve the infinitesimal causal structure. The Adjoint action in the group of symmetries is shown to encode precisely the effects of changes of inertial frames in the theory. Then we define observer-independent and static causal structures, formalising what it means for future-directedness to be observer-independent. These notions highly constrain the action of the group on spacetime: its Lie algebra must admit an Adjoint-invariant convex cone, and thus the group must satisfy Segal's assumptions. All the backgrounds of supergravity or string theories which have a causal Killing field satisfy these assumptions.

Our second purpose is to classify the causal structures of spacetimes admitting Killing symmetries in terms of properties of their groups of motions. It turns out we can fully characterise observer-independent causal structures, but also static metrics, only in terms of particular invariant cones in the symmetry groups. These essential features are simply encoded in the Adjoint action of the symmetry group. It becomes clear why de-Sitter cannot be static, since its group of symmetries does not admit an Adjoint-invariant cone. These results are directly applicable to any spacetime with at least one symmetry, and are particularly simple for product spaces. Our formalism also allows us to deal with non-static spacetimes in an original way: we describe the horizons of homogeneous spacetimes as horizons in their group of symmetries, and are then able to see the effect of changes of inertial frames and changes of observers on the horizon structure. Horizons are completely defined group theoretically.

Our third purpose is to show that the classical notion of positive energy –as defined by a causal Killing field on spacetime–, is related to the quantum mechanical one –by a one-side bounded operator on a Hilbert space–. We define a Dirac procedure for the universal algebra of a symmetry group, and describe the corresponding quantum theory in a relativistic invariant manner. It turns out the Hilbert space can be given a causal structure stemming from the locally allowed Hamiltonians on spacetime, so that staticity of spacetime metrics is equivalent to the existence of global time in quantum theory. The physically inequivalent Hamiltonians for stationary observers in spacetime are classified as orbits under the Adjoint action of the symmetry group. We then use some results of unitary highest weight representations to show that the existence of an Adjoint invariant convex cone in the group of symmetries goes hand in hand with the existence of positive energy operators for the quantum states. Thus one can classify symmetry groups on the physical grounds of staticity of the causal structure, and the existence of a positive energy. It also follows that for any static spacetime with a simple group of symmetries, there exists a highest weight representation, and hence the usual Fock space can be built.

We have tried to make the presentation both axiomatic and self-contained –which unfortunately is incompatible with brevity–, and illustrate our results with examples. The plan of the paper is as follows.

In Sec.2 we briefly review some salient properties of group actions on manifolds, and fix some notation used throughout the article. Our point of view is to describe Killing vector fields of spacetimes as elements of the Lie algebra of the group of motions. Sec.3 introduces infinitesimal causal structures on manifolds and Lie groups, summarises some of Segal's results [15] regarding bi-invariant convex cones, and raises questions about the physical interpretation of these results. To answer these, in Sec.4, we discuss in detail the link between the symmetries of a physical theory with a spacetime interpretation, and the causal structure of the spacetime. We highlight the difference between changes of inertial frames and changes of observers, and show that

for certain observables, the Adjoint action in the group of motions represents in the Lie algebra the effect of changes of inertial frames. Particular attention is given to the rôle of the universal algebra in terms of defining observables, and to the classification of inequivalent observables as orbits under the Adjoint action. These results enable us to give in Sec.5 a detailed account of the physical consequences of the existence of bi-invariant cones in a group of motions. In a series of Lemmas, we show how these relate to observer-independent causal structures and static metrics on spacetimes. As an illustration we give a detailed analysis of the  $AdS_2$  spacetime. Our method provides a classification of the inequivalent times of  $AdS_2$  in 3 types, which is relevant for the black hole or matrix model interpretation of this spacetime [20–22]. We then focus in Sec.6 on homogeneous spacetimes which admit horizons. We lift the horizons to loci in the group itself and describe how and whether some horizons are related by changes of inertial frames. Then, in Sec.7, we apply the Dirac procedure to the universal algebra, and the group structure allows us to define time evolution in a relativistic invariant way. The notion of observer-dependence allows us to classify the inequivalent Hamiltonians of  $AdS_2$ . We then formally establish a relation between the existence of positive energy operators on a space of states defined by a unitary representation space of the group of motions, and bi-invariant convex cones in the Lie group. We conclude in Sec.8.

## 2 Some properties of group actions

**Definition 1** *A Lie group  $G$  acts on a manifold  $\mathcal{M}$  if there exists a smooth homomorphism from  $G$  into the group of diffeomorphisms of  $\mathcal{M}$ . In other words there exists a smooth map  $\Gamma : G \times \mathcal{M} \rightarrow \mathcal{M}$ ,  $(g, p) \mapsto g.p$ , such that for all  $p \in \mathcal{M}$ , for all  $g_1, g_2 \in G$ , and with  $e$  denoting the identity element of  $G$ , we have:*

$$e.p = p \quad \text{and} \quad (g_1 g_2).p = g_1.(g_2.p)$$

The action in this case is called a *left* action. For fixed  $p \in \mathcal{M}$  we denote by  $\mu_p : G \rightarrow \mathcal{M}$ ,  $g \mapsto g.p$  the restriction of  $\Gamma$  to  $G$ , and for fixed  $g \in G$  by  $\nu_g : \mathcal{M} \rightarrow \mathcal{M}$ ,  $p \mapsto g.p$  the restriction of  $\Gamma$  to  $\mathcal{M}$ .

For fixed  $p \in \mathcal{M}$  the image of  $\mu_p$  is called the  $G$ -orbit of  $p$ , and the group action is said *transitive* if for one (and hence for all)  $p \in \mathcal{M}$ ,  $\mu_p$  is onto. We call  $H_p = \{g \in G / \nu_g(p) = p\}$  the stabiliser subgroup of a point  $p \in \mathcal{M}$ . The  $H_p$  are Lie-subgroups of  $G$ , and there are all conjugate if the action is transitive: for all  $g \in G$ ,  $H_{g.p} = gH_pg^{-1}$ . In this case, with  $H$  the stabiliser group of a chosen point  $p \in \mathcal{M}$ , the bijection  $gH \mapsto g.p$  between the coset space  $G/H$  and  $\mathcal{M}$  (locally compact and connected) can be made into a diffeomorphism [23, pp.110–115] and we call  $\mathcal{M} \simeq G/H$  a *homogeneous* space. This diffeomorphism chooses a particular point in  $\mathcal{M}$  which corresponds to  $eH$ .

By definition the map  $g \mapsto \nu_g$  is a homomorphism of  $G$  into the group of diffeomorphisms of  $\mathcal{M}$ . Its kernel  $N$  is a closed normal subgroup of  $G$ . If  $N = \{e\}$  (or is a discrete subgroup of  $G$ ) the action is said *effective* (or respectively almost effective). Given  $G$  acting on  $\mathcal{M}$ , the Lie group  $G/N$  acts effectively on  $\mathcal{M}$ , so that there is no restriction in considering such actions. If  $G$  is simple, the action is necessarily almost effective.

We denote by  $R_g$  and  $L_g$  right and left translations in  $G$  by an element  $g \in G$ , and by  $F * (x)$  the differential of a function  $F$  at  $x$ . When convenient we shall represent

tangent vectors  $V_x$  at  $x$  by a one-parameter curves  $[f(t)]$  such that  $f(0) = x$  and  $f'(0) = V_x$ .

The two following results which relate the Lie algebra of a Lie transformation group to special vector fields of the manifold it acts on, are standard, and will be used repeatedly throughout the article. They derive from Lie's first and second fundamental theorems.

**Theorem 1** *Let  $G$  a Lie group with Lie algebra  $\mathfrak{g}$  act on a manifold  $\mathcal{M}$ . Then the map defined by*

$$\phi : A \in \mathfrak{g} \longmapsto [X^A : p \mapsto X_p^A \equiv \mu_p * (e)A = [\exp tA, p]] \quad (1)$$

*is a Lie algebra anti-homomorphism from  $\mathfrak{g}$  into the set of vector fields of  $\mathcal{M}$ :  $\phi$  is linear and such that  $\phi([A, B]) = -[\phi(A), \phi(B)]$ , or equivalently*

$$X^{[A, B]} = -[X^A, X^B].$$

*Proof:* For fixed  $p \in \mathcal{M}$ ,  $\mu_p$  is a differentiable map between  $G$  and  $\mathcal{M}$ . For  $A \in \mathfrak{g}$  define the right invariant vector field  $\tilde{A}_g = R_g * (e)A$  on  $G$ . Using  $\mu_p \circ R_g = \mu_{g.p}$  together with (1), we get

$$\begin{aligned} \mu_p * (g)\tilde{A}_g &= \mu_{g.p} * (e)A \equiv X_{g.p}^A \\ &= X_{\mu_p(g)}^A. \end{aligned}$$

Thus the vector fields  $\tilde{A}$  and  $X^A$  are  $\mu_p$ -related, and it follows [23, p.24] that  $\mu_p * (g)[\tilde{A}, \tilde{B}]_g = [X^A, X^B]_{g.p}$ . With the convention that the Lie bracket in  $\mathfrak{g}$  is defined as that of left-invariant vector fields at the identity, we have  $[\tilde{A}, \tilde{B}]_e = -[A, B]$ . Although the  $X^A$  are here defined on the  $G$ -orbit of  $p$  only, this reasoning can be done at all  $p$ , and the theorem follows.  $\square$

For a right action, one gets a homomorphism; however when considering isometries of a spacetime, left actions appear more naturally, so we shall keep the minus sign. It is clear that in this case  $-\phi$  defines a Lie algebra homomorphism.

For  $\mathcal{M}$  a connected manifold equipped with a non-degenerate metric tensor field  $h$ , we call Killing vector fields of  $(\mathcal{M}, h)$  the Killing fields which are complete in  $\mathcal{M}$ —so that their flows generate one-parameter groups of diffeomorphisms of  $\mathcal{M}$ , and which close under Lie bracket—so that their flows define a group—. Physically this means that the integral curves of the Killing fields should stay in the spacetime, and that the set of induced transformations (sometimes called Killing motions) should be stable under composition. For maximally extended spacetimes, it should not be a problem, and the cases when Killing fields are not complete can be attributed to a bad choice of coordinates. With this in mind, Theorem 1 has a converse:

**Theorem 2** *Let  $\mathcal{M}$  a (connected) manifold equipped with a metric tensor field  $h$ . Then there exists a Lie group  $G$  with Lie algebra  $\mathfrak{g}$  which acts on  $\mathcal{M}$ , such that the Killing vector fields of  $\mathcal{M}$  are precisely the images of elements of  $\mathfrak{g}$  by the anti-homomorphism defined by (1).*

*Proof:* One shows that the transformation group of  $\mathcal{M}$  generated by the flows of the Killing fields of  $(\mathcal{M}, h)$  can be given a suitable topology such that it is Lie group. Its

Lie algebra is (anti-)isomorphic to that defined by the Killing fields, and this defines a map  $\phi$  as in (1).  $\square$

As a consequence, the vector fields  $X^A$  in Theorem 1 will also be called Killing vector fields, though without referring to a particular metric on  $\mathcal{M}$ . From now on, we shall always think of Killing vector fields of a spacetime  $(\mathcal{M}, h)$  as images of an element of a Lie algebra.

Theorem 1 applies to any manifold  $\mathcal{M}$ , and not just Lorentzian spacetimes of relativity. For example, a symmetry group may act on the cotangent bundle of a spacetime (phase-space), or even just on a vector space of solutions to a differential equation. Even when  $\mathcal{M}$  can be interpreted as a “spacetime”, it need not be equipped with a metric such that  $G$ -motions are isometries. For instance the Newton-Cartan spacetime [24] and the Newton-Hooke spacetimes [25], though they admit transitive actions of the Galilei and Newton-Hooke groups respectively, do not admit invariant non-degenerate metrics. Given any action of a group on a manifold, one can construct vector fields in  $\mathcal{M}$  invariant under  $G$ -motion in the sense that  $X_{g,p}^A = \mu_{g,p} * A = (\mu_p \circ R_g) * A = \mu_p * (g) \tilde{A}_g$ . The  $\nu_g : p \rightarrow g.p$  are mere diffeomorphisms, and the vector fields  $X^A$  their generators. The symmetries become physically relevant if they correspond to symmetries of particular equations of motion: the Galilei and Newton-Hooke transforms leave invariant respectively Newton’s equations in flat space or their analogue with a cosmological constant term [25]. When one considers a Lorentzian manifold in Theorem 2, the Killing motions are isometries of the metric tensor field, with the additional physical meaning that the structure of spacetime or more precisely the geodesic equations describing the free fall of particles, do not change along these orbits. Thus the diffeomorphisms  $\nu_g$  correspond to changes of inertial frames. We shall come back to this in detail in Sec.4.

In the general setting of Theorem 1, the properties of the physics one can do on a homogeneous space  $G/H$  depend on the existence or non-existence of  $G$ -invariant tensor fields on  $G/H$ , which in turn is related to properties of the Adjoint action in  $G$ . The latter will also play an important rôle in any spacetime  $(\mathcal{M}, h)$  with symmetry group  $G$ .

**Definition 2** Let  $G$  a Lie group with Lie algebra  $\mathfrak{g}$ . The Adjoint representation<sup>2</sup> of  $G$  is the following group homomorphism:

$$\begin{aligned} \text{Ad} : G &\longrightarrow GL(\mathfrak{g}) \\ g &\longmapsto [\text{Ad}_g : B \mapsto (L_g \circ R_{g^{-1}}) * (e)B = [g(\exp tB)g^{-1}]]. \end{aligned}$$

The Adjoint action of a subgroup  $H$  of  $G$  on  $\mathfrak{g}$  is the group of automorphisms  $\text{Ad}_h$  for  $h \in H$ .

This is not to be confused with the adjoint representation of  $\mathfrak{g}$ , which the following Lie algebra homomorphism:

$$\begin{aligned} \text{ad} : \mathfrak{g} &\longrightarrow gl(\mathfrak{g}) \\ A &\longmapsto [\text{ad}_A : B \mapsto [A, B]]. \end{aligned}$$

In Lie theory, a homogeneous space  $G/H$  of  $G$ , as opposed to a mere coset space, is usually defined by a splitting of  $\mathfrak{g}$  into  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  where  $\mathfrak{h}$  is the Lie algebra of the

<sup>2</sup>In the physics literature, where  $G$  is often a matrix Lie group and hence left and right translations are equal to their differentials, one has  $\text{Ad}_g B = g B g^{-1}$ .

subgroup  $H$  so that  $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ , and where in addition  $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$ . We will call *reductive* the homogeneous spaces  $G/H$  such that the Lie algebra  $\mathfrak{h}$  admits an  $Ad_H$ -invariant complement  $\mathfrak{m}$  in  $\mathfrak{g}$ , so that the decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is  $Ad_H$ -invariant.<sup>3</sup> Then the tangent space of  $G/H$  at  $eH$  is identified with  $\mathfrak{m}$  and the linear isotropy representation of  $H$  with the restriction of the Adjoint action of  $H$  on  $\mathfrak{m}$ . It then follows that there is a one-to-one correspondence between  $G$ -invariant tensor fields on  $G/H$  (that is, invariant under the  $\nu_g$ ), and  $Ad_H$ -invariant tensor fields on  $\mathfrak{m}$ . For example, this is how one defines left invariant metrics on homogeneous spacetimes  $\mathcal{M} \simeq G/H$  in the setting of Theorem 1, so that the physics one can do on such spacetimes depends on the existence of such  $Ad_H$ -invariant metrics. These facts are widely known but their details are not important to us, since we will take a different approach. We will consider spacetimes which do not necessarily admit a transitive group action, and the Adjoint action will encode properties of their causal structure.

### 3 Segal causal structures and bi-invariant cones

We review Segal's approach [15] to causal structures on manifolds and particularly Lie groups. Generally speaking, the approach lacks precise physical motivations, and we try to remedy to this first in Sec.4.

A global causal structure on a manifold  $\mathcal{M}$  is simply a partial ordering of its points, ie a transitive antisymmetric relation, often written  $x \prec y$ . Antisymmetry excludes the possibility of time travel. For example in a time-orientable spacetime of general relativity one can define:  $x \prec y$  if and only if there exists a curve from  $x$  to  $y$  whose tangent vector at every point lies in the cone of future-directed time-like or light-like vectors. This is a partial order as long as there are no (non-trivial) closed timelike or null future-directed curves. The existence of such a property depends on global features of spacetimes, some of which are encoded in the symmetries. Segal had the idea of defining local causal structures, which together with “well-behaved” global symmetries, might imply the existence of global causal structures. With  $T_p\mathcal{M}$  denoting the tangent space of  $\mathcal{M}$  at  $p$ , one can define the following as in [15]:

**Definition 3** *An infinitesimal causal structure on a manifold  $\mathcal{M}$  is a smooth assignment at every point  $p \in \mathcal{M}$  of a non-trivial closed convex cone  $C_p \subset T_p\mathcal{M}$  which is pointed, i.e. such that  $C_p \cap -C_p = \{0\}$ . We shall call  $(\mathcal{M}, C_p)$  a Segal structure.*

We say a cone  $C_p \subset T_p\mathcal{M}$  is *Einsteinian* if it contains a basis of  $T_p\mathcal{M}$ , or equivalently, if its interior  $Int(C_p)$  (with respect to the topology on  $T_p\mathcal{M}$ ) is non-empty. Such cones yield the Segal structures of physical interest. Given  $(\mathcal{M}, C_p)$ , we will call a curve  $t \mapsto \gamma(t)$  in  $\mathcal{M}$  *causal* if its tangent vectors at each point lie in the cone at that point. Of course the reversed curve  $t \mapsto \gamma(-t)$  is *not* causal unless  $\gamma$  is trivial. Causal vectors and causal vector fields on  $(\mathcal{M}, C_p)$  are defined similarly. Then the relation  $x \prec y$  if there exists a causal curve from  $x$  to  $y$  defines a transitive relation on  $\mathcal{M}$ .  $\mathcal{M}$  need not be equipped with a metric –and in fact Segal never does so– although a sufficiently smooth metric tensor field can provide us with a Segal structure. This notion can of course be applied to Lie groups, where in addition some Segal structures can be related to the group structure: a (non-trivial closed) pointed convex cone in the Lie algebra  $\mathfrak{g}$  of  $G$  can be right (or left) translated to every point in  $G$ , and hence define a right (or left) invariant Segal structure  $(G, C_g)$ , with  $C_g = R_g * (e)C_e$  (or  $L_g * (e)C_e$ ).

<sup>3</sup>In the general case we just have  $Ad_{\exp \mathfrak{h}} \mathfrak{m} \subset \mathfrak{m}$  and not  $Ad_H \mathfrak{m} \subset \mathfrak{m}$ .

Then  $C_e \in \mathfrak{g}$  is invariant under  $Ad_G$  if and only if it defines a *bi-invariant* structure, in which case we shall simply denote it by  $(G, C_e)$ . It seems natural then to define causally preserving maps:

**Definition 4** *A map between two manifolds equipped with Segal structures is said to be causally preserving if it sends causal curves to causal curves, or equivalently if the differential of the map sends the cone at each point into the cone at the image point.*

One can then consider causally preserving actions of groups on manifolds, by requiring that for all  $g \in G$ , the  $\nu_g$  be causally preserving diffeomorphisms of  $(\mathcal{M}, C_p)$ . In the case of homogeneous manifolds, the Segal structure on  $\mathcal{M} \simeq G/H$  is then invariant under left translations on cosets, and for reductive spaces, as with left-invariant tensor fields, it is uniquely determined by an  $Ad_H$ -invariant cone in  $\mathfrak{m}$  which is  $\nu_g$ -translated in  $G/H$ . We explain in the next section why the symmetries of many physical theories induce causally preserving group actions, as is the case for connected groups of motions of Lorentzian spacetimes.

For extra mathematical simplicity, one can focus on conal structures on Lie groups which are both left and right invariant, and analyse how they map to homogeneous spaces. It will turn out that these stringent assumptions are in fact satisfied as soon as the group  $G$  acts on a static spacetime  $(\mathcal{M}, h)$ . Paneitz [16] introduces bi-invariant structures by noticing the following: since  $G$  and  $\mathcal{M}$  can both be given Segal structures, one can ask that both structures be related by the full group action, by requiring that  $\Gamma$  in Definition 1 be a causally preserving map from  $G \times \mathcal{M}$  to  $\mathcal{M}$ . We shall call such an action *fully causally preserving*. Explicitly,  $\Gamma^*$  sends the cones  $C_g \oplus C_p$  at  $(g, p)$  into the cones  $C_{g,p}$  at  $g.p$ . We will show in Sec.5 that the existence of a bi-invariant structure on  $G$  is necessary in this case.

Segal, Paneitz [16, 17], Vinberg [14] and others (see [18]) have classified various Lie groups admitting bi-invariant convex cones.

**Theorem 3** [15] *A simple real Lie group  $G$  with maximal compact subgroup  $K$  admits an  $Ad_G$ -invariant (non-trivial closed pointed Einsteinian) convex cone in  $\mathfrak{g}$  if and only if  $G/K$  is a hermitian symmetric space.*

*Proof:* This follows from a theorem of Kostant which states that a real finite dimensional vector space  $V$  acted upon by a (connected) semisimple Lie group  $G$  with maximal compact subgroup  $K$  admits a non-trivial  $G$ -action invariant cone if and only if it admits a non-trivial  $K$ -action invariant vector (see [15]).  $\square$

There are four families of irreducible (non-compact) hermitian symmetric algebras, plus two exceptional ones. Their corresponding simple Lie groups  $G$  and maximal compact subgroups  $K$  are given in Table 1. In fact for  $G$  simple,  $G/K$  is hermitian symmetric if and only if  $K$  has a one-dimensional centre. The reader is referred to [23, Chapters 8 & 9] for notations and proofs. The spaces  $G/K$  in Table 1 are also called Cartan classical domains. We shall come back to these in Sec.7 through representation theory, and their relevance to positive energy. The exceptional low dimensional isomorphisms between some of the groups in this classification imply that some of these hermitian spaces are identical. Note also that  $\mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{so}(2, 1) \simeq \mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{su}(1, 1)$ , and  $SO(2, 2)/SO(2) \times SO(2)$  is not in the list since it is not irreducible but equal to  $(SO(2, 1)/SO(2)) \times (SO(2, 1)/SO(2))$ .

The simple compact groups are not in this classification, but the compact groups with non-trivial centres admit Adjoint invariant cones. For example, the set of posi-

| $G$                                 | $K$                   |
|-------------------------------------|-----------------------|
| $Sp(2n, \mathbb{R})$ ( $n \geq 1$ ) | $U(n)$                |
| $SU(p, q)$ ( $p \geq q \geq 1$ )    | $S(U(p) \times U(q))$ |
| $SO^*(2n)$ ( $n \geq 3$ )           | $U(n)$                |
| $SO(2, n)$ ( $n \geq 3$ )           | $SO(2) \times SO(n)$  |
| $E_{6(-14)}$                        | $SO(2) \times SO(10)$ |
| $E_{7(-25)}$                        | $E_6 \times U(1)$     |

Table 1: Irreducible hermitian symmetric spaces  $G/K$ 

tive matrices (multiplied by  $\sqrt{-1}$ ) defines an Adjoint-invariant pointed cone in the Lie algebra of the unitary group  $U(n)$ . In quantum mechanics, this cone is interpreted as the set of mixed states or their associated probability functionals [26].

In the case of semi-simple algebras, one can have direct sums of invariant cones in the simple summands, with some cones being possibly trivial, so that one of the summands must be in Table 1. Of course, the cones might not be pointed nor Einsteinian. The Poincaré group  $E(1, n)$ , symmetry group of Minkowski space  $\mathbb{E}^{1, n}$  equipped with its flat metric, admits a bi-invariant cone, but is non-simple. For direct products of Lie groups, if one group admits an Adjoint-invariant cone, then so does the product. This will be the case for the symmetry groups  $SO(2, n) \times SO(q)$  of  $AdS_{n+1} \times S^{q-1}$  spacetimes for example. Also, all the (connected) Lie groups admitting bi-invariant Lorentzian metrics evidently possess such cones. Such groups arise for example in the classification of maximally supersymmetric vacua of supergravity in six dimensions [27].

Paneitz shows in [16, Theorem 16.5] that the universal covers of the classical groups  $G$  in Table 1 admit *global* causal structures which do not possess closed causal curves: these are defined by bi-invariant Segal structures  $(G, C_e)$  stemming from particular Adjoint-invariant cones  $C_e \subset \mathfrak{g}$ . This important achievement seems to be the only a posteriori justification to why bi-invariant cones in the Lie algebra of a group of motion are physically relevant. It is well-known that closed causal curves can appear when one takes the quotient of a space, hence this result cannot be extended to all coset spaces of these groups. Typical examples are provided by anti-de-Sitter space or other quotients of its universal cover by discrete or continuous orbits. More recently, it was shown that the Gödel supergravity solution of [1], which has closed time-like curves, can be obtained as a reduction of a six dimensional plane-wave itself isomorphic to a Lorentzian Lie group [27, and references therein]. On the other hand, de-Sitter space equipped with its usual metric does not admit closed causal curves, though its symmetry group  $SO(1, n+1)$  does not admit an invariant cone. These simple remarks suggest that bi-invariant cones may be more fundamentally related to staticity of spacetime metrics rather than to the absence of closed causal curves. Indeed, we will not worry about the global causal structure from now, but explain why Adjoint-invariant cones are necessary whenever the notion of future-directedness in a Segal structure  $(\mathcal{M}, C_p)$  is required to be observer-independent.

Before going into the mathematical details, it is necessary to clarify the physical relevance of group actions on manifolds and the rôle of the Adjoint action in the symmetry groups.

## 4 Physical interpretation

### 4.1 Changes of inertial frames in $\mathcal{M}$

We now justify, from a physical point of view, why it is expected that the action of a symmetry group of a theory with spacetime interpretation, induces causally preserving diffeomorphisms of an infinitesimal causal structure on the spacetime. The space of states of the theory is unspecified: it is a subset of a real vector space, describing particles, fields or extended objects. Some properties of the theory, such as notions of what may possibly be allowed versus what may definitely not, are encoded in a Segal structure  $(\mathcal{M}, C_p)$ . This only determines locally at each event all possible future events, and certainly does not specify the whole theory. We assume that for all  $p \in \mathcal{M}$ , any causal vector  $V_p \in C_p$  “partially describes” at least one acceptable state of the theory (for example  $V_p$  is the momentum at  $p$  of a one-particle state, or of a 0-brane . . .). Mathematically, there exists for each  $p \in \mathcal{M}$  a subset  $\Omega_p$  of the set of physical states, and a map  $\Psi_p$  from  $\Omega_p$  to the future cone  $C_p \subset T_p \mathcal{M}$ , which is onto. This assumption does violate Heisenberg’s uncertainty principle.<sup>4</sup> Moreover, the  $\Psi_p$  are assumed to be real linear (we have to include mixed states).

The weakest *principle of symmetry* [28] says that a symmetry of the theory should take any acceptable state of the theory to another acceptable state of the theory. We consider here *active* symmetries, since passive ones, such as gauge symmetries, act trivially on the physical states.<sup>5</sup> A stronger principle says that any two states which are related by the action of a symmetry are *physically equivalent* (but nonetheless different). The latter generally implies that the action on the states is unitary, but we need not make this assumption here. For our purpose, we shall just assume that it takes a state “partially described” by a causal vector at a point, to a different state which is also “partially described” by causal vectors at some other points. Calling  $\Theta(g)$  the action of a particular symmetry on the states of the theory, given any  $p \in \mathcal{M}$ , there exists at least one  $q \in \mathcal{M}$  such that the following diagram holds:

$$\begin{array}{ccc} \Omega_p & \xrightarrow{\Theta(g)} & \Omega_q \\ \Psi_p \downarrow & & \downarrow \Psi_q \\ C_p & & C_q \end{array}$$

We need to specify the physical assumptions further to turn this into a commutative diagram and show that we can choose  $q$  uniquely given  $p$ . The *principle of indistinguishability*, which stems from the principle of relativity, says that states which are indistinguishable for an observer, get mapped under a symmetry, to equivalent states which remain indistinguishable for an equivalent observer. In other words, if any two physical states are “partially described” by the same vector at  $p \in \mathcal{M}$ , they get mapped by  $\Theta(g)$  to physical states such that there exists a point  $q \in \mathcal{M}$  at which they are “partially described” by the same vector. Otherwise,  $\Theta(g)$  distinguishes the states. If this is not true for a very general theory with different objects, we can restrict the theory to subsets of states such that it become true. Thus for  $p \in \mathcal{M}$  fixed, there exists  $q \in \mathcal{M}$  such that  $\Theta(g)(\text{Ker}(\Psi_p)) \subset \text{Ker}(\Psi_q)$ . Similarly, there exist  $r \in \mathcal{M}$  such

<sup>4</sup>We are not saying that the states in  $\Omega_p$  have position  $p \in \mathcal{M}$  and momentum  $V_p \in C_p$ . For example,  $\Psi_p$  can be the expectation value of the momentum of a wave function at the point  $p \in \mathcal{M}$ .

<sup>5</sup>Of course gauge theories are included in this discussion, the gauge freedom being factored out for clarity. On the other hand, the “symmetries” of string theory which relate physical states of theories with different background spacetimes (T-duality for example) are not considered here.

that  $\Theta(g^{-1})(\text{Ker}(\Psi_q)) \subset \text{Ker}(\Psi_r)$  and thus  $\text{Ker}(\Psi_p) \subset \text{Ker}(\Psi_r)$ . The last assumption is that there are sufficiently many physical states to distinguish the points of spacetime<sup>6</sup>, by which we mean that for any two different points  $p', r' \in \mathcal{M}$ , there exists a vector  $v$  (a real linear combination of physical states), such that  $v \in \text{Ker}(\Psi_{p'})$  but  $v \notin \text{Ker}(\Psi_{r'})$ . This implies that  $p \equiv r$ , so that  $\Theta(g)(\text{Ker}(\Psi_p)) = \text{Ker}(\Psi_q)$ , and hence also that  $q \in \mathcal{M}$  is unique given  $p$ . Thus we can define  $\vartheta(g) : \mathcal{M} \rightarrow \mathcal{M}$ ,  $p \mapsto \vartheta(g)p \equiv q$ . We have postulated or physically motivated the existence of the following commutative diagram:

$$\begin{array}{ccc} \Omega_p & \xrightarrow{\Theta(g)} & \Omega_q \\ \Psi_p \downarrow & & \downarrow \Psi_q \\ C_p & \xrightarrow{\theta(g)} & C_q \end{array} \quad (2)$$

which defines the map  $\theta(g) : C_p \rightarrow C_q$ . Since the elements of  $C_p$  can in fact also be interpreted as classical velocities of particular spacetime observers, we must have, for all  $V_p \in C_p$ ,  $\theta(g)V_p = \vartheta(g)*(p)V_p$ .

Standard arguments [30] show that, if the space of states is a vector space, the map  $g \mapsto \Theta(g)$  defines a representation of the symmetry group  $G$  on the space of states (possibly after a central extension of  $G$  and a modification of  $\Theta$  by a phase [31]). In all cases  $g \mapsto \Theta(g)$  is a homomorphism up to factors of indistinguishability in the space of states. One can then easily derive that for all  $g_1, g_2 \in G$ ,  $p \in \mathcal{M}$  and  $u \in \Omega_p$ ,

$$\theta(g_1)\theta(g_2)\Psi_p(u) = \theta(g_1g_2)\Psi_p(u),$$

and the maps  $g \mapsto \theta(g)$  and  $g \mapsto \vartheta(g)$  define group homomorphisms. If further the symmetry group  $G$  is a Lie group, and the maps  $\Psi_p$  are smooth, letting  $\vartheta(g) \equiv \nu_g$  so that  $\nu_g* \equiv \theta(g)$ , we have the following:

**Lemma 1** *Let  $(\mathcal{M}, C_p)$  a Segal structure. Let  $G$  a symmetry group of a physical theory on  $\mathcal{M}$  satisfying the assumptions above. Then the action of  $G$  on the physical states induces, upon restriction to a subset of physical states, an action on spacetime which sends any causal vector to a causal vector. There exists a homomorphism  $g \mapsto \nu_g$  from  $G$  into the diffeomorphisms of  $\mathcal{M}$ , such that:*

$$\forall g \in G, \quad \forall p \in \mathcal{M}, \quad \nu_g*(p)(C_p) = C_{g.p}. \quad (3)$$

The induced homomorphism from  $G$  into the causally preserving diffeomorphisms of  $(\mathcal{M}, C_p)$  is rarely onto or injective. For example, the local conformal diffeomorphisms of a Lorentzian manifold –which are indeed causally preserving– are seldom all induced by symmetries of the full physical theory. Typically conformal invariance is broken in theories with massive particles. On the other hand, in many theories, multiplication by a global phase factor or global charge conjugation on the states do not change velocities so that these symmetries induce the trivial action on spacetime. The same is true for gauge symmetries.

Of course equation (3) follows directly if we simply consider a theory of causally related events or causal curves on  $(\mathcal{M}, C_p)$ . The explanation given above is far more general. From now on, when we talk of a theory of causal curves, we shall have in mind that

<sup>6</sup>This is related to the existence of (spatially) localised states as constructed in [29], which are taken to physically equivalent ones by  $\Theta(g)$ . Momentum eigen-states such as  $e^{-ip_\mu x^\mu}$  will on the other hand yield  $p^\mu$  irrespective of the position.

it is in fact a “partial description” of a more general theory satisfying the suitable assumptions. Lemma 1 justifies using Segal’s assumption of causally preserving group actions, on spacetimes which are not necessarily homogeneous.

**Definition 5** *In the setting of Lemma 1, or given a causally preserving group action on a Segal structure  $(\mathcal{M}, C_p)$ , we call changes of inertial frames or inertial transforms, the spacetime diffeomorphisms  $\nu_g$ , for each  $g \in G$ .*

This is a spacetime concept: it may well be that a symmetry acts non-trivially on the states of the theory, but trivially on spacetime. The  $\nu_g$  relate a frame at  $p \in \mathcal{M}$  to a frame  $g.p \in \mathcal{M}$ . In Minkowski space, these generally correspond to the Poincaré transforms in the component of the identity of the full Poincaré group, although some symmetries might be broken or enhanced in particular theories. The  $\nu_g*$  are then restricted Lorentz transforms which relate tangent vectors at different points of spacetime. Time inversion is never an inertial transform, since choosing a Segal structure fixes a time orientation, whereas space inversion might be.

So far “inertial” does not refer to the invariance of a particular universal law or equation of motion, since we have simply dealt with causality relations in a manifold. We now consider Lorentzian spacetimes, so that Killing symmetries yield transforms  $\nu_g$  which preserve the geodesic structure of spacetime, and hence are inertial in the usual relativistic sense.

For completeness we show that given a Lorentzian spacetime as in Theorem 2, the group  $G$  can be taken so that the  $\nu_g$  be causally preserving. This is elementary. Let  $(\mathcal{M}, h)$  a time-oriented manifold with Lorentzian metric tensor field  $h$ , and equip  $\mathcal{M}$  with the Segal structure stemming from  $h$  and a time direction.

The fact that  $X$  is a Killing field of  $(\mathcal{M}, h)$  is usually seen infinitesimally as  $\mathcal{L}_X h = 0$ : the Lie derivative of  $h$  along the direction of the vector field  $X$  vanishes. From a global point of view, this means that the flow of  $X$  is an isometry: the differential of the map which sends a point  $p$  to another point along an integral curve of  $X$  through  $p$  preserves the metric inner product. This flow corresponds to a diffeomorphism  $\nu_g : \mathcal{M} \rightarrow \mathcal{M}$  for some  $g \in G$ , where  $G$  is the Lie group constructed as in Theorem 2 from the Killing vector fields of  $(\mathcal{M}, h)$ : indeed, for  $A \in \mathfrak{g}$ , the Killing vector field  $\phi(A) \equiv X^A$ , has flow  $\exp tA.p$ . Therefore any Killing motion is equivalent to a global left action on  $\mathcal{M}$  of a group element  $g = \exp tA$ , for some  $t \in \mathbb{R}$ , and we can take  $G$  connected. Thus for all  $g \in G$ , for all  $p \in \mathcal{M}$ ,  $\nu_g*(p) : T_p \mathcal{M} \rightarrow T_{g.p} \mathcal{M}$  is such that:

$$\forall X, Y \in T_p \mathcal{M}, \quad h_p(X, Y) = h_{g.p}(\nu_g*(p)X, \nu_g*(p)Y) \quad (4)$$

Equivalently, the pull-back of  $h$  under  $\nu_g$  is equal to  $h$ . This implies the following

**Lemma 2** *Let  $(\mathcal{M}, h)$  a time-oriented Lorentzian manifold equipped with its corresponding Segal structure  $(\mathcal{M}, C_p)$ , and let  $G$  a connected Lie group whose action on  $(\mathcal{M}, h)$  represents its Killing motions. Then for all  $g \in G$ , the  $\nu_g : \mathcal{M} \rightarrow \mathcal{M}$ ,  $p \mapsto g.p$ , are causally preserving diffeomorphisms of  $(\mathcal{M}, C_p)$ .*

*Proof:* Calling  $T$  a (non-vanishing) future directed vector field in  $\mathcal{M}$ , define

$$C'_p = \{V \in T_p \mathcal{M} / h_p(V, V) < 0, h_p(V, T_p) < 0\}. \quad (5)$$

For  $t \mapsto g(t)$  a continuous path from  $g(0) = e$  to  $g(1) = g \in G$  and  $h_p(V, V) < 0$ , (4) implies that  $t \mapsto h_{g(t).p}(\nu_{g(t)}*(p)V, T_{g(t).p})$  never vanishes and thus remains strictly

negative. Since  $C_p = \overline{C'_p}$ ,  $\nu_g * (p)C_p = C_{g.p}$  for all  $g \in G$ .  $\square$

The equivalent of parallel transport, as an isometry, along orbits of a group of motion, is the differential  $\nu_g *$ . More importantly,  $\nu_g * (p)X_p$  for an observer at  $g.p$  is physically equivalent to  $X_p$  for an observer at  $p$ : the  $\nu_g$  relate different equivalent states of a theory, one as seen by observers in a frame at  $p \in \mathcal{M}$ , to another one as seen by observers in a frame at  $g.p \in \mathcal{M}$ .

If the symmetries of a physical theory induce via Lemma 1 a transitive action on the spacetime  $\mathcal{M} \simeq G/H$ , we get the principal fibre bundle formulation  $(G, H, \mathcal{M} \simeq G/H)$ : the isometry group  $G$  is the bundle of inertial frames, the fibres  $H$  the homogeneous symmetries which represent inertially related frames at one point, the base manifold the spacetime  $\mathcal{M}$ , and the changes of inertial frames are the left translations in  $G$ . These act on spacetime as well as on the fibers of course.

As we will see next, the notion of inertial transforms on spacetime observables can be described quite simply and elegantly in the Lie algebra  $\mathfrak{g}$  of the symmetry group.

## 4.2 Changes of inertial frames as Adjoint actions in $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$

The observables of a physical theory are usually defined as operators on the space of states, and generally constitute a real vector space. One also requires the existence of real-valued functionals associated to each observable, which send a state to the expectation value of the observable on this state. These also constitute a real vector space. Note that in quantum theories, real-linearity of the observable functionals (in their argument) together with the complex vector space structure of the Hilbert space of pure states, requires introducing mixed states.<sup>7</sup> For both mathematical simplicity and physical relevance, the (operator) observables must be bounded, or equivalently must have a finite operator norm. The existence of an associative product law between observables, though refuted by many on physical grounds (see [32]), is postulated in classical mechanics (Poisson algebras, see [33] for example), in quantum mechanics [34], quantum field theory, and by extension also in string theories (bounded self-adjoint operators). We shall thus assume it. Note that this does not mean that the set of observables is stable under that product.

In a theory whose symmetry generators constitute a Lie algebra, the *universal algebra*<sup>8</sup> of this Lie algebra has the required structure. Moreover, its structure becomes necessary if we make the following assumption: the generators of the symmetries of a theory should define observables whose commutators correspond to the observable associated to the Lie bracket of the generators. Then indeed, by definition (see [23, p.90] or [35, Chapter 2]), the commutator law on the observables is that stemming from the universal algebra. This is valid for any theory with symmetries, and not just theories with a spacetime interpretation. We shall come back to this in Sec.7 with the Dirac quantization procedure. In this section, we shall motivate the rôle of the universal algebra from a spacetime point of view, by building differential operator observables out of its elements. We will show that the generalised Adjoint action corresponds to the effect of changes of inertial frames on those particular observables.

For  $p \in \mathcal{M}$ , we denote by  $\mathcal{T}(\mathfrak{g}) \equiv \mathbb{R}.1 \bigoplus_{n \geq 1} \otimes^n \mathfrak{g}$  and  $\mathcal{T}(T_p \mathcal{M}) \equiv \mathbb{R}.1 \bigoplus_{n \geq 1} \otimes^n T_p \mathcal{M}$  the algebras of contravariant tensors of any rank on  $\mathfrak{g}$  and  $T_p \mathcal{M}$  respectively, equipped

<sup>7</sup>In standard notation,  $[\lvert v \rangle \mapsto \langle v \lvert A \rvert v \rangle]$  is neither  $\mathbb{C}$  nor  $\mathbb{R}$ -linear in  $\lvert v \rangle$ , whereas  $[\lvert v \rangle \langle v \rvert \mapsto \text{Tr}(A \lvert v \rangle \langle v \rvert)]$  is  $\mathbb{R}$ -linear in  $\lvert v \rangle \langle v \rvert$ .

<sup>8</sup>also called the universal enveloping algebra or the enveloping algebra

with tensor multiplication and unit elements 1. For  $p \in \mathcal{M}$  and  $g \in G$ , the linear maps  $\mu_p * (e)$ ,  $Ad_g$  and  $\nu_g * (p)$  are uniquely extended to morphisms of/between these tensor algebras, using tensor multiplication. For a homogeneous manifold, the effect of a change of frame on a contravariant tensor at a point can be visualised as an Adjoint action its group of motions:

**Lemma 3** *Let  $G$  a Lie group with Lie algebra  $\mathfrak{g}$  act transitively on a spacetime  $\mathcal{M}$ . Then for all  $p \in \mathcal{M}$  fixed,*

$$\begin{aligned}\mu_p * (e) : \mathcal{T}(\mathfrak{g}) &\longrightarrow \mathcal{T}(T_p \mathcal{M}) \\ T &\longmapsto \mu_p * (e)T\end{aligned}$$

defines a surjective map. For all  $g \in G$ , we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{T}(\mathfrak{g}) & \xrightarrow{Ad_g} & \mathcal{T}(\mathfrak{g}) \\ \mu_p * (e) \downarrow & & \downarrow \mu_{g.p} * (e) \\ \mathcal{T}(T_p \mathcal{M}) & \xrightarrow{\nu_g * (p)} & \mathcal{T}(T_{g.p} \mathcal{M}) \end{array} \quad (6)$$

so that the tensor-generalised Adjoint action  $Ad_g : \mathcal{T}(\mathfrak{g}) \rightarrow \mathcal{T}(\mathfrak{g})$  represents at  $e \in G$ , the effect of a spacetime inertial transformation  $\nu_g$  on a contravariant tensor at  $p \in \mathcal{M}$ .

*Proof:* That  $\mu_p * (e)$  is onto follows from the fact that it is onto from  $\mathfrak{g}$  to  $T_p \mathcal{M}$  when  $\mathcal{M} \simeq G/H$ . Let  $B \in \mathfrak{g}$ . We have  $\mu_p * (e)B \equiv X_p^B \equiv [\exp tB.p]$  so that

$$\begin{aligned}\nu_g * (p)\mu_p * (e)B &\equiv \nu_g * (p)X^B \equiv [g \exp tB.p] = [g \exp tB g^{-1}g.p] \\ &= \mu_{g.p} * (e)[g \exp tB g^{-1}] \\ &\equiv \mu_{g.p} * (e)(Ad_g B)\end{aligned} \quad (7)$$

Diagram (6) follows by generalisation, with  $Ad_g(A \otimes B) \equiv Ad_g(A) \otimes Ad_g(B)$  and  $Ad_g(1) \equiv 1$  and similarly for  $\nu_g * (p)$ .  $\square$

Diagram (6) however is only defined at each fixed point  $p \in \mathcal{M}$ , not even on neighbourhoods of a point, and thus is only relevant for *local* effects of changes of inertial frames. Tensor fields on  $\mathcal{M}$  do not generally correspond to single elements in the tensor algebra  $\mathcal{T}(\mathfrak{g})$ .

For a fixed vector  $B \in \mathfrak{g}$ , the generalisation of diagram (6) to all  $p \in \mathcal{M}$  naturally defines, through equation (1), the (Killing) field  $X^B$  and we get the following commutative diagram:

$$\begin{array}{ccc} B \in \mathfrak{g} & \xrightarrow{Ad_g} & Ad_g B \in \mathfrak{g} \\ \phi \downarrow & & \downarrow \phi \\ X^B & \xrightarrow{\nu_g *} & X^{Ad_g B} \end{array} \quad (8)$$

Thus the global effect on a Killing vector field, of a change of inertial frame on spacetime, is particularly simple:

$$\begin{aligned}\forall p \in \mathcal{M}, \quad \nu_g * (p)X_p^B &= X_{g.p}^{Ad_g B} \\ \text{or} \quad [p \mapsto X_p^B] &\text{ becomes } [p \mapsto X_p^{Ad_g B}]\end{aligned} \quad (9)$$

We now think of the elements of  $\mathfrak{g}$  as defining some observables of a physical theory. Typically their associated Killing fields can be thought of as differential operators acting on a set of functions on  $\mathcal{M}$ . Diagram (8) shows that these observables behave simply under inertial transforms, which suggests they should be useful in the physical interpretation. We then define an associative product between these observables, and under the assumption that the commutator of two observables in a Lie algebra be given by their Lie bracket, we have to consider universal algebras. We denote by  $\mathcal{U}(\mathfrak{g})$  the universal algebra of  $\mathfrak{g}$  and  $\mathcal{U}(\phi(\mathfrak{g}))$  the universal algebra of the Lie algebra of Killing vector fields on  $\mathcal{M}$  defined in Theorem 1.

**Proposition 1** *Let  $\mathcal{M}$  a spacetime acted upon almost effectively by its group of symmetries  $G$ . With  $\tilde{\phi}$  and  $\widetilde{Ad}$  denoting the natural extensions of the maps  $\phi$  and  $Ad$  to  $\mathcal{U}(\mathfrak{g})$ , we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{U}(\mathfrak{g}) & \xrightarrow{\widetilde{Ad}_g} & \mathcal{U}(\mathfrak{g}) \\ -\tilde{\phi} \downarrow & & \downarrow -\tilde{\phi} \\ \mathcal{U}(\phi(\mathfrak{g})) & \xrightarrow{\widetilde{\nu_g}^*} & \mathcal{U}(\phi(\mathfrak{g})) \end{array} \quad (10)$$

so that the generalised Adjoint action on elements of the universal algebra  $\mathcal{U}(\mathfrak{g})$  represents the effect of changes of inertial frames on particular spacetime observables on  $\mathcal{M}$ .

Two observables in  $\mathcal{U}(\mathfrak{g})$  are physically equivalent (for different observers) if and only if they are related by an Adjoint action up to scale.

*Proof:* This follows from (8), Theorem 1, and the definitions of the maps involved. Since the action is almost effective,  $-\phi$  of Theorem 1 defines a Lie algebra isomorphism between  $\mathfrak{g}$  and  $\phi(\mathfrak{g})$ . For  $A, B \in \mathfrak{g}$

$$(-\tilde{\phi})(A \otimes B - B \otimes A - [A, B]) = X^A \otimes X^B - X^B \otimes X^A - [X^A, X^B] \quad (11)$$

so that elements in the two-sided ideal of  $\mathcal{T}(\mathfrak{g})$  spanned by  $A \otimes B - B \otimes A - [A, B]$ , are mapped into the two-sided ideal of  $\mathcal{T}(\phi(\mathfrak{g}))$  spanned by the r.h.s of (11). Hence  $(-\tilde{\phi})$  is an isomorphism from  $\mathcal{U}(\mathfrak{g})$  to  $\mathcal{U}(\phi(\mathfrak{g}))$ .

Diagram (8) extends trivially to the tensor algebras  $\mathcal{T}(\mathfrak{g})$  and  $\mathcal{T}(\phi(\mathfrak{g}))$ , so that, for  $A, B \in \mathfrak{g}$  and  $g \in G$ , we have:

$$\begin{aligned} & \widetilde{\nu_g}^* \circ (-\tilde{\phi})(A \otimes B - B \otimes A - [A, B]) \\ &= \widetilde{\nu_g}^*(X^A \otimes X^B - X^B \otimes X^A + X^{[A, B]}) \\ &= X^{Ad_g A} \otimes X^{Ad_g B} - X^{Ad_g B} \otimes X^{Ad_g A} + X^{Ad_g [A, B]} \\ &= (-\tilde{\phi})(Ad_g A \otimes Ad_g B - Ad_g B \otimes Ad_g A - Ad_g [A, B]) \\ &= (-\tilde{\phi}) \circ \widetilde{Ad}_g (A \otimes B - B \otimes A - [A, B]) \end{aligned}$$

which implies that (8) extends to diagram (10).

The last comment follows directly from Definition 5. We note again that the physical observables will generally correspond to a sub-vector-space of  $\mathcal{U}(\mathfrak{g})$ , which should be Adjoint invariant.  $\square$

Thus the physically inequivalent Killing vector fields of a spacetime are classified or

labelled by the (projective) orbits in  $\mathfrak{g}$  under the Adjoint action of  $G$ . However, when one considers a particular observer at  $p \in \mathcal{M}$ , one can only quotient by  $Ad_{H_p}$ . In Sec.7, we will show that projective orbits in  $\mathfrak{g}$  also classify the inequivalent quantum theory Hamiltonians of stationary observers in  $\mathcal{M}$ . We will also study the case of  $AdS_2$  in Sec.5.3, and show that the classification of the inequivalent times of this spacetime is clarified in this group theoretic picture.

In fact, given any smooth map  $U : \mathcal{M} \rightarrow \mathfrak{g}$ ,  $[p \mapsto U(p)]$ , the map  $[p \mapsto \mu_p * (e)U(p)]$  defines a smooth vector field on  $\mathcal{M}$  (and similarly for contravariant tensors of other ranks). One can easily derive that under a change of frame  $\nu_g$ ,

$$\begin{aligned} U : \mathcal{M} &\longrightarrow \mathfrak{g} & \text{becomes} & \quad g(U) : \mathcal{M} \longrightarrow \mathfrak{g} \\ p &\longmapsto A(p) & p &\longmapsto Ad_g A(g^{-1}p) \end{aligned}$$

and thus represent in  $\mathfrak{g}$  the effect of a change of frame on any vector field on a homogeneous spacetime, and similarly for contravariant tensor fields. The question is then how big the set of relevant observables should be.  $\mathcal{U}(\mathfrak{g})$  and  $\mathcal{U}(\phi(\mathfrak{g}))$  are infinite dimensional Lie algebras with countable basis (Poincaré-Birkhoff-Witt theorem [35, Theorem 2.1.11]) whereas allowing for any contravariant tensor field will generically lead to algebras with uncountable basis. We will not consider this further.

It is important to note that, whereas the elements of the tensor algebra  $\mathcal{T}(\phi(\mathfrak{g}))$  defined (sums of) contravariant tensor fields on  $\mathcal{M}$  built out of the Killing fields, the elements of the universal algebra  $\mathcal{U}(\phi(\mathfrak{g}))$  define particular differential *operators* on  $\mathcal{M}$ , whose product is given by composition. Proposition 1 shows the relevance of the universal algebra of the symmetry group of a theory, in terms of particular spacetime observables including the Killing field differential operators on  $\mathcal{M}$ . Precisely, under assumptions we made clear,  $\mathcal{U}(\mathfrak{g})$  should contain a minimal set of observables. The spacetime need not be homogeneous of course. This also shows that there is no need to postulate the effect of changes of inertial frames in the universal algebra: the Adjoint action follows directly from the definition of the spacetime changes of inertial frames. From this, we will derive in Sec.7.2 the changes of inertial frame in the quantum theories.

The results of this subsection, especially Diagram (8) and Proposition 1, give a precise setting to address the question of the physical meaning of Adjoint-invariant structures in the group of symmetries of a general theory. We show next that, whereas the Adjoint action represents at  $e \in G$  the effect of changes of inertial frames in  $\mathcal{M}$ ,  $Ad_G$ -invariance is in fact related to invariance under changes of *observers* in  $\mathcal{M}$ . For this, we first define what we mean by changes of observers.

### 4.3 Changes of observers

We explained in Sec.4.1 why the action of a symmetry group  $G$  on a spacetime  $\mathcal{M}$  should send causal curves to causal curves, and called the induced spacetime diffeomorphisms changes of inertial frames. We now ask ourselves what it means to require that any *fixed* curve which is causal for an observer remain causal for distant observers. Generally speaking this question is not well defined: one has to compare tangent vectors at different points of spacetime, and this requires making extra assumptions, since there is no unique way of defining common future directions at different events. Similar difficulties occur when defining a notion of simultaneity of events. One requirement is that the definition of invariance of “future-directedness” under change of observers,

must itself be invariant under the changes of inertial frames. Moreover, since only observers related by an inertial transform can observe equivalent states of the theory in their respective frames, we can only define observer (in)dependence for such physically equivalent observers. It will turn out that observer-independent causal structures require the existence of causal Killing fields. Although this may suggest our approach to causality is too simple, we will show in Sec.7 that it is related to the existence of global times and globally valid Hamiltonians in quantum theories, which vindicates its physical relevance.

Let  $(\mathcal{M}, C_p)$  a Segal structure, and assume for the moment that the future cones are given by a Lorentzian metric  $h$  and a time-orientation. The symmetries of  $\mathcal{M}$  are those of Lemma 2. We consider a theory of causal curves or causal velocities on  $(\mathcal{M}, C_p)$ , which we may view as a “partial description” of a general theory. We want to compare a velocity  $V_p \in C_p$  which is causal for an observer at  $p \in \mathcal{M}$ , to a possible future direction for an equivalent observer at  $g.p \in \mathcal{M}$ . In a way, we need to postulate the covariance rules. The diffeomorphisms  $\nu_g$  map  $V_p \in C_p$  to  $\nu_g * (p)V_p \in C_{g.p}$ , but up to now this was interpreted in diagram (2) as mapping a state represented by a vector at  $p$  to a different but physically equivalent state represented by a vector at  $g.p$ . We now re-interpret the active transform  $V_p \mapsto \nu_g * (p)V_p$  as a passive transform, or a change of spacetime coordinates. This is clear when the group element  $g$  is in the stabiliser of  $p \in \mathcal{M}$ , so that  $g.p$  is a Lorentz boosted or rotated observer, and we extend it for all  $g \in G$ .

To make the present scenario clear, we define observables with respect to the observers in the spacetime, and dissociate them from the states of the theory.<sup>9</sup> In a theory of causal curves in  $(\mathcal{M}, h)$ , a tangent vector  $X_p \in T_p\mathcal{M}$  defines the functional  $V_p \mapsto h_p(X_p, V_p)$ , which we can call a velocity observable for an observer at  $p$ .  $X_p$  is causal if and only if the expectation value of its observable is negative for any state with  $V_p \in C_p$ . Such an observable measures future directedness at  $p$ . Reciprocally, a state has a causal velocity at  $p$  if and only if the expectation values of all causal velocity observables  $h_p(X_p, \cdot)$ , with  $X_p \in C_p$ , are negative.<sup>10</sup>

Then there are two equivalent ways of defining the question whether a vector which is causal for an observer at a point, remain causal for different equivalent observers: one postulates a way of extending either the states, or the observables, to every equivalent spacetime point. Proposition 1 suggests Killing fields should be used, when it is possible, to extend our local velocity observables to every equivalent point.

Consider an observer at the event  $p$  in a homogeneous spacetime  $\mathcal{M} \simeq G/H$ , moving along a causal curve with tangent  $X_p$  in the interior of  $C_p$ . There exists  $A \in \mathfrak{g}$  such that  $X_p = X_p^A$ , though  $A$  is not generally unique. Since  $X^A$  remains causal in a neighborhood  $\mathcal{N}$  of  $p$ , the integral curves  $\exp tA.x$  for  $x \in \mathcal{N}$  are causal for all  $t \in \mathbb{R}$ . These are rarely geodesics of  $(\mathcal{M}, h)$ , but the observers moving along them are static with respect to each other. (They are called stationary observers, and will play an important rôle in the quantum theories.) Hence for any  $A \in \mathfrak{g}$  such that  $X_p^A = X_p$ , the Killing vector field  $X^A$  is a good candidate for a definition of future direction extended to a neighbourhood  $\mathcal{N}$ . In other words  $X^A$  defines a Killing observable which extends the velocity observable defined by  $X_p$  to any spacetime point

<sup>9</sup>In fact we only consider the expectation value of the observables, so that nothing is really measured in the quantum sense.

<sup>10</sup>Equation (4) can be interpreted either as a change of coordinates on the representatives of both states and observables, or as an inertial transform to different but physically equivalent states and observables.

$q \in \mathcal{M}$  by  $V_q \mapsto h_q(X_q^A, V_q)$ . Moreover, the expectation values of this observable are constant along the geodesics of  $(\mathcal{M}, h)$  and hence it yields a constant of motion for states represented by geodesics.

We say that a state  $V_p \in C_p$  for an observer at  $p \in \mathcal{M}$  with future  $X_p \in C_p$  remains causal for an equivalent observer at  $g.p$  if and only if there exists a Killing observable  $X^A$  which extends  $X_p$  to  $g.p$  and is such that  $V_p$  is causal at  $g.p$ , in other words:

$$h_{g,p}(X_{g,p}^A, \nu_g * (p)V_p) \leq 0, \quad (12)$$

where  $V_p \mapsto \nu_g * (p)V_p$  is just a change of coordinates from  $p$  to  $g.p$ . Requiring this for all states  $V_p \in C_p$  and all  $g \in G$ , given  $X_p$  fixed, simply implies that  $X_{g,p}^A \in C_{g,p}$  for all  $g.p \in \mathcal{M}$ : the observer-independence of the future-directness of states is equivalent to the existence of Killing velocity observables which are causal on a given  $G$ -orbit, or equivalently for homogeneous spacetimes, of causal Killing fields. One can say that such an observable  $X^A$  defines a common clock for the observers on the  $G$ -orbit of  $p \in \mathcal{M}$ .

One obtains the same conclusion by extending a state with a Killing field, and letting the observer at  $g.p$  measure it with any local causal velocity observable  $X_{g,p} \in C_{g,p}$ . Thus we make the following

**Definition 6** *Let a Lie group  $G$  act on a manifold  $\mathcal{M}$ . A Segal structure  $(\mathcal{M}, C_p)$  is called observer-independent if, for any point  $p \in \mathcal{M}$  and any causal vector  $X_p \in C_p$ , there exists a (Killing) field  $X^A$  as defined by (1) which is causal on the  $G$ -orbit of  $p$  and such that  $X_p^A = X_p$ .*

When the Segal structure  $(\mathcal{M}, C_p)$  is Einsteinian, its observer-independence implies the homogeneity of  $\mathcal{M}$ , since for all  $p \in \mathcal{M}$ , the  $\mu_p * (e)$  must be onto. Thus the Killing fields of the definition are causal on the whole spacetime. Minkowski space  $\mathbb{E}^{1,n-1}$  and Anti-de-Sitter space  $AdS_n$  with their usual metric Segal structures, are common examples of such spaces.

**Definition 7** *Let a Lie group  $G$  act on a manifold  $\mathcal{M}$ . A Segal structure  $(\mathcal{M}, C_p)$  is said to be static<sup>11</sup> if there exists a (Killing) field  $X^A$  as defined by (1) such that for all  $p \in \mathcal{M}$ ,  $X_p^A \in \text{Int}(C_p)$ .*

This definition only makes sense for Einsteinian Segal structures, which are the physically relevant cases. Of course, if a Segal structure  $(\mathcal{M}, C_p)$  is observer-independent or static with respect to a certain symmetry group  $G$ , it is so for any bigger symmetry group having  $G$  as a subgroup. For Minkowski space, it suffices to consider the abelian subgroup of the Poincaré group generated by the translations. Note that spacetimes  $(\mathcal{M}, h)$  which have a static Killing field but are not homogeneous cannot have an observer-independent Segal structure. Four dimensional examples can be found in [36, Chapter 16]. There are also examples of spacetimes which admit a causal Killing field (null in fact), are homogeneous spaces, but whose causal structure is not observer-independent. The Kaigorodov spacetimes [37, 38] and the homogeneous plane-waves are such spaces. We give conditions which relate staticity and observer-independence in the next section.

Suppose now the group action induces causally preserving diffeomorphisms of  $(\mathcal{M}, C_p)$ . Clearly equation (3) implies that a Killing field  $X^A$  of  $(\mathcal{M}, C_p)$  is causal

<sup>11</sup>In fact, stationary would be more meaningful. Note that the Killing fields of Lorentzian spacetimes  $(\mathcal{M}, h)$  are assumed to have complete orbits.

if and only if for any  $g \in G$ ,  $\nu_g * X^A$  is also a causal Killing field. Thus Definitions 6 and 7 are indeed invariant under changes of inertial frames.

Since the  $C_p$  are convex cones, the set of causal Killing fields of  $(\mathcal{M}, C_p)$  is also a convex cone. The Lie algebra of the Killing fields, as a finite-dimensional vector space, has a standard topology, for which this cone is closed. Its antecedent under  $\phi$  of (1) is a closed convex cone in  $\mathfrak{g}$ . Using diagram (8), the invariance of the causal Killing fields under changes of inertial frames implies that the corresponding cone in  $\mathfrak{g}$  is invariant under the Adjoint action of  $G$ . We will be more precise in the next section.

Our discussion of the rôle of symmetries in a general physical theory have justified Segal's assumption to consider group actions which preserve infinitesimal causal structures. We have just shown that observer-independence and staticity of the causal structures require the existence of Adjoint invariant closed convex cones in the Lie algebra of the symmetry group, and thus justified physically Segal's second assumption. In fact these assumptions hold in a spacetime  $(\mathcal{M}, h)$  whenever there is a causal Killing field.

## 5 Adjoint invariant cones and static spacetimes

We precisely formulate observer-independence and staticity of Segal structures in terms of properties of the group action. We go from the mathematical results to the physical ones, and end the section with a detailed discussion on two-dimensional Anti-de-Sitter space.

### 5.1 Fully causally preserving group actions

**Lemma 4** *Let  $G$  a Lie group with Lie algebra  $\mathfrak{g}$  act on manifold  $\mathcal{M}$  non-trivially but not necessarily transitively. If there exists Segal structures  $(G, C_g)$  and  $(\mathcal{M}, C_p)$  such that the action  $\Gamma : G \times \mathcal{M} \rightarrow \mathcal{M}$  is fully causally preserving, then there exists a non-trivial (proper) closed convex cone  $C_e \subset \mathfrak{g}$  which is stable under  $Ad_G$ . If the action is almost effective,  $\tilde{C}_e \cap -\tilde{C}_e = \{0\}$ , and further if  $G$  is simple,  $C_e$  is Einsteinian.*

*Proof:* If  $\Gamma$  is causally preserving, then so are  $\mu_p : G \rightarrow \mathcal{M}$  and  $\nu_g : \mathcal{M} \rightarrow \mathcal{M}$  for all  $p \in \mathcal{M}$  and  $g \in G$ . Thus for each  $p \in \mathcal{M}$ ,  $C_e^p \equiv \{A \in \mathfrak{g} / \mu_p * (e)A \in C_p\}$  is a closed convex cone containing  $C_e$ . The  $\nu_g$  are causally preserving diffeomorphisms of  $\mathcal{M}$ :

$$V_p \in C_p \Leftrightarrow \nu_g * (p)V_p \in C_{g.p}$$

so that using (9) we have:

$$\begin{aligned} A \in C_e^p &\Leftrightarrow X_p^A C_p \\ &\Leftrightarrow \nu_g * (p)X_p^A = X_{g.p}^{Ad_g A} \in C_{g.p} \\ &\Leftrightarrow Ad_g A \in C_e^{g.p} \end{aligned} \tag{13}$$

In other words  $C_e^{g.p} = Ad_g C_e^p$ . Define:

$$\begin{aligned} \tilde{C}_e &\equiv \bigcap_{p \in \mathcal{M}} C_e^p \equiv \bigcap_{p \in \mathcal{M}} \{A \in \mathfrak{g} / \mu_p * (e)A \in C_p\} \\ &\equiv \{A \in \mathfrak{g} / \forall p \in \mathcal{M}, X_p^A \in C_p\} \end{aligned} \tag{14}$$

so that for all  $g \in G$ ,

$$\tilde{C}_e = \bigcap_{p \in \mathcal{M}} C_e^{g \cdot p} = \bigcap_{p \in \mathcal{M}} Ad_g C_e^p = Ad_g \tilde{C}_e.$$

$\tilde{C}_e$  is a closed convex cone in  $\mathfrak{g}$  containing  $C_e$ , and it is invariant under  $Ad_G$ . In addition, we have:

$$\tilde{C}_e \cap -\tilde{C}_e = \bigcap_{p \in \mathcal{M}} (C_e^p \cap -C_e^p) = \bigcap_{p \in \mathcal{M}} \text{Ker}(\mu_p * (e)), \quad (15)$$

and  $\tilde{C}_e \cap -\tilde{C}_e \neq \mathfrak{g}$  since there exist  $p \in \mathcal{M}$  such that  $\mu_p$  is not trivial. Hence  $\tilde{C}_e$  is a non-trivial proper subcone of  $\mathfrak{g}$ . Moreover, if the action is almost effective, the map  $\phi$  in Theorem 1 is injective, so (15) implies that  $\tilde{C}_e$  is pointed. In all cases,  $\tilde{C}_e \cap -\tilde{C}_e$  is a vector space in  $\mathfrak{g}$  which is  $Ad_G$ -invariant, so it is an ideal of  $\mathfrak{g}$ . The same is true for the vector span of  $\tilde{C}_e$  in  $\mathfrak{g}$ . Thus if  $\mathfrak{g}$  simple,  $\tilde{C}_e$  is pointed and Einsteinian.  $\square$

Note that if the cones  $C_p$  are not pointed, as in a Newtonian theory on  $\mathcal{M}$ , (15) does not hold and  $\tilde{C}_e$  is not necessarily pointed (but it is still (pointed) Einsteinian whenever  $G$  is simple). Lemma 4 has the following converse:

**Lemma 5** *Let  $G$  a Lie group admitting a non-trivial  $Ad_G$ -invariant pointed closed convex cone  $C_e \subset \mathfrak{g}$ , and let  $G$  act transitively on a manifold  $\mathcal{M} \simeq G/H$ . Suppose there exists  $p \in \mathcal{M}$  such that the closure of  $\mu_p * (e)C_e$  in  $T_p \mathcal{M}$  is a (non trivial) pointed cone. Then  $G$  and  $\mathcal{M}$  can be given Segal structures such that the action  $\Gamma : G \times \mathcal{M} \rightarrow \mathcal{M}$  is fully causally preserving.*

*Calling  $\mathfrak{h}$  the Lie algebra of  $H = H_p$ , if  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is an  $Ad_H$ -invariant split of  $\mathfrak{g}$ , this condition is equivalent to  $\pi_{\mathfrak{m}}(C_e)$  is pointed and closed in  $\mathfrak{m}$ , where  $\pi_{\mathfrak{m}}$  the projection of  $\mathfrak{g}$  on  $\mathfrak{m}$ .*

*Proof:* Let  $C_g = R_g * C_e = L_g * Ad_{g^{-1}} C_e = L_g * C_e$  define a (bi-invariant) Segal structure on  $G$ . Call  $C_p \equiv \mu_p * (e)C_e \equiv \{X_p^A / A \in C_e\}$ . We first show that the map  $g \cdot p \mapsto \nu_g * (p)C_p \equiv C_{g \cdot p}$  defines, up to closure, a Segal structure on  $\mathcal{M}$ .

For  $h \in G$  such that  $hg \cdot p = g \cdot p$ , using (9) and  $Ad_G$ -invariance of  $C_e$ , we have:

$$\begin{aligned} C_{g \cdot p} &\equiv \nu_g * (p)C_p = \{X_{g \cdot p}^{Ad_g A} / A \in C_e\} = \{X_{g \cdot p}^B / B \in C_e\} \quad (16) \\ \text{so that } C_{hg \cdot p} &= \{X_{hg \cdot p}^{Ad_h B} / B \in C_e\} = \{X_{g \cdot p}^{Ad_g A} / A \in C_e\} \\ &= C_{g \cdot p} \end{aligned}$$

Since the  $\nu_g$  are diffeomorphisms and  $\overline{C_p}$  is pointed by hypothesis,  $g \cdot p \mapsto \overline{C_{g \cdot p}}$  is a smooth assignment on  $\mathcal{M}$  of (closed) pointed cones; hence  $(\mathcal{M}, \overline{C_{g \cdot p}})$  is a Segal structure. By definition the  $\nu_g$  are causally preserving. For all  $p \in \mathcal{M}$  and  $g \in G$ , using  $\mu_p \circ R_g = \mu_{g \cdot p}$  and (16),

$$\begin{aligned} \mu_p * (g)C_g &= \mu_p * (g)R_g * (e) = \mu_{g \cdot p} * (e)C_e \\ &= \{X_{g \cdot p}^A / A \in C_e\} = C_{g \cdot p} \subset \overline{C_{g \cdot p}} \end{aligned} \quad (17)$$

so that the  $\mu_p$  are causally preserving. Now for any  $(g, q) \in G \times \mathcal{M}$ , as  $C_g = L_g * C_e$  and  $C_q = \mu_q * (e)C_e$ , any element of  $C_g \oplus C_q$  is represented by a curve  $[(g \exp tB, \exp tA \cdot p)]$  with  $A, B \in C_e$ , and

$$\begin{aligned} \Gamma * (g, p)[(g \exp tB, \exp tA \cdot p)] &\equiv [\Gamma(g \exp tB, \exp tA \cdot p)] = [(g \exp tB \exp tA) \cdot p] \\ &= [\mu_p(g \exp tB \exp tA)] = \mu_p * (g)[g \exp tB \exp tA] \end{aligned}$$

From (17) it suffices to show that  $[g \exp tB \exp tA] \in C_g$  or equivalently that  $[\exp tB \exp tA] \in C_e$ . This follows from (see [23] p.96 for example)

$$\exp tB \exp tA = \exp(t(B + A) + \frac{t^2}{2}[B, A] + O(t^3))$$

which implies that  $[\exp tB \exp tA] = B + A \in C_e$ . By continuity we get  $\Gamma(C_g \oplus \overline{C_q}) \subset \overline{C_q}$ , hence  $\Gamma$  is causally preserving.

Taking  $H = H_p$  the stabiliser subgroup of  $p \in \mathcal{M} \simeq G/H$ ,  $\mathfrak{m}$  is isomorphic to the tangent space of  $G/H$  at  $eH$ , so to  $T_p\mathcal{M}$ , under:

$$\begin{aligned} \mathfrak{m} &\rightarrow T_{eH}G/H \\ A &\mapsto [\exp tA eH] \end{aligned} \tag{18}$$

Via the diffeomorphism  $G/H \simeq \mathcal{M}$ , (18) corresponds to the restriction to  $\mathfrak{m}$  of  $\mu_p * (e) : \mathfrak{g} \rightarrow T_p\mathcal{M}$ . This restriction is then an isomorphism, ie  $\mathfrak{h} = \text{Ker}(\mu_p * (e))$ , and  $\mu_p * (e)C_e = \mu_p * (e)(\pi_{\mathfrak{m}}(C_e))$  is pointed and closed if and only if  $\pi_{\mathfrak{m}}(C_e)$  is pointed and closed in the topology of  $\mathfrak{m}$ .  $\square$

Note that if in addition  $C_e$  is Einsteinian,  $\mu_p * (e)C_e$  spans  $T_p\mathcal{M}$  so the Segal structure defined on  $\mathcal{M}$  is Einsteinian.

## 5.2 Observer-independent and static causal structures

We now relate the two previous Lemmas on fully preserving group actions to the physical formalism of Sec.4. All the results derive quite simply, and some may seem to be just restatements for the thorough reader. We show how observer-independence and staticity of spacetime causal structures drastically reduce the variety of allowed spacetime symmetry groups. We hope to make the physical relevance clear.

### 5.2.1 Observer-independent causal structures

**Lemma 6** *Let  $G$  a Lie group act almost effectively on a manifold  $\mathcal{M}$  with a Segal structure  $(\mathcal{M}, C_p)$ . Suppose that for all  $g \in G$ , the  $\nu_g$  are causally preserving diffeomorphisms of  $(\mathcal{M}, C_p)$ . Then  $(\mathcal{M}, C_p)$  admits a causal (Killing) field as defined by (1) if and only if there exists a bi-invariant Segal structure  $(G, C_e)$  such that the action  $\Gamma : G \times \mathcal{M} \rightarrow \mathcal{M}$  is fully causally preserving.*

*Proof:*  $\Rightarrow$  As in the proof of Lemma 4,  $C_e \equiv \{A \in \mathfrak{g} / \forall p \in \mathcal{M}, X_p^A \in C_p\}$  defines an  $Ad_G$ -invariant pointed closed convex cone. It is non-empty by hypothesis, and we can define the bi-invariant Segal structure  $(G, C_e)$ . Using the end of the proof of Lemma 5, we show that  $\Gamma : G \times \mathcal{M} \rightarrow \mathcal{M}$  is fully causally preserving.

$\Leftarrow$  Let  $A \in C_e$ , so that for all  $p \in \mathcal{M}$ ,  $X_p^A \equiv \mu_p * (e)A \in C_p$  by hypothesis. Since the action is almost effective,  $X^A$  is non-trivial.  $\square$

When  $\mathcal{M}$  is homogeneous, we just need a condition at a point:

**Corollary 1** *Let  $(\mathcal{M}, C_p)$  and  $G$  as in Lemma 6. If  $\mathcal{M}$  is homogeneous, then  $(\mathcal{M}, C_p)$  admits a causal (Killing) field if and only if there exists an Adjoint-invariant (closed pointed convex) cone  $C_e \subset \mathfrak{g}$  and a point  $p \in \mathcal{M}$  such that  $\mu_p * (e)C_e \subset C_p$  and is non-trivial.*

We can now straightforwardly characterise observer-independent Segal structures:

**Proposition 2** *Let  $G$  a Lie group act almost effectively and transitively on a manifold  $\mathcal{M}$  with Segal structure  $(\mathcal{M}, C_p)$ . Suppose that for all  $g \in G$ , the  $\nu_g$  are causally preserving diffeomorphisms of  $(\mathcal{M}, C_p)$ . Then  $(\mathcal{M}, C_p)$  is observer independent if and only if the following conditions are satisfied:*

- (i) *there exists a bi-invariant Segal structure  $(G, C_e)$  such that action  $\Gamma : G \times \mathcal{M} \rightarrow \mathcal{M}$  is fully causally preserving,*
- (ii) *there exists  $p \in \mathcal{M}$  such that  $\mu_p * (e)C_e = C_p$ .*

*Proof:*  $\Rightarrow$  (i) is the same as in Lemma 6, so that (ii) follows by Definition 6.

$\Leftarrow$  Let  $q \in \mathcal{M}$ ,  $V_q \in C_q$ . There exist  $g \in G$  and  $V_p \in C_p$  such that  $g \cdot p = q$  and  $\nu_g * (p)V_p = V_q$ . Then there exists  $A \in C_e$  such that  $X_p^A = V_p$  extends  $V_p$  to a causal (Killing) field. Then  $X_q^{Ad_g A}$  extends  $V_q$  to a causal (Killing) field.  $\square$

The transitivity condition is superfluous when  $(\mathcal{M}, C_p)$  is Einsteinian. The following version of Proposition 3 will be more useful (see Sec.5.3):

**Corollary 2** *Let  $(\mathcal{M}, h)$  a time-oriented Lorentzian spacetime, and let  $G$  a (connected) Lie group with Lie algebra  $\mathfrak{g}$  represent the Killing motions of  $(\mathcal{M}, h)$  as in Lemma 2. Then the metric Segal structure  $(\mathcal{M}, C_p)$  is observer independent if and only if the following conditions are satisfied:*

- (i) *there exists  $C_e$  a pointed closed convex cone in  $\mathfrak{g}$  which is  $Ad_G$ -invariant,*
- (ii) *there exists  $p \in \mathcal{M}$ ,  $\mu_p * (e)C_e = C_p$ .*

### 5.2.2 Static spacetimes

From now the spacetime Segal structures  $(\mathcal{M}, C_p)$  are Einsteinian so that Definition 7 makes sense.

**Proposition 3** *Let  $G$  a Lie group act almost effectively on a manifold  $\mathcal{M}$  with a Segal structure  $(\mathcal{M}, C_p)$ . Suppose that for all  $g \in G$ , the  $\nu_g$  are causally preserving diffeomorphisms of  $(\mathcal{M}, C_p)$ . Then  $(\mathcal{M}, C_p)$  is static if and only if the following conditions are satisfied:*

- (i) *there exists a bi-invariant Segal structure  $(G, C_e)$  such that action  $\Gamma : G \times \mathcal{M} \rightarrow \mathcal{M}$  is fully causally preserving,*
- (ii)  $C_e \cap (\bigcap_{p \in \mathcal{M}} \{A \in \mathfrak{g} / X_p^A \in \text{Int}(C_p)\}) \neq \emptyset$ .

*Proof:* This is straightforward since  $\bigcap_{p \in \mathcal{M}} \{A \in \mathfrak{g} / X_p^A \in \text{Int}(C_p)\} \neq \emptyset$  is just Definition 7

$\Rightarrow$   $C_e \equiv \{A \in \mathfrak{g} / \forall p \in \mathcal{M}, X_p^A \in C_p\}$  defines a non-empty  $Ad_G$ -invariant pointed closed convex cone. Using the end of the proof of Lemma 5, we show that  $\Gamma : G \times \mathcal{M} \rightarrow \mathcal{M}$  is fully causally preserving.

$\Leftarrow$  Any element  $B \in C_e \cap (\bigcap_{p \in \mathcal{M}} \{A \in \mathfrak{g} / X_p^A \in \text{Int}(C_p)\})$  defines a causal (Killing) vector field  $X^B$  on  $(\mathcal{M}, C_p)$ .  $\square$

$SO(1, n)^+$  –the identity component of  $SO(1, n)^-$ , as we noticed earlier, does not admit an Adjoint invariant cone. However by Lemma 2 it acts by causally preserving diffeomorphisms on de Sitter space  $dS_n = SO(1, n)^+ / SO(1, n-1)^+$  equipped with its usual Einstein metric. Propositions 2 and 3 then imply that the causal structure of de-Sitter space can be neither observer-independent nor static.

In contrast with de-Sitter space, there can exist bi-invariant cones in the group of motions which do not satisfy all the requirements for staticity or the existence of a

causal Killing field. The fact that the action  $\Gamma$  be fully causally preserving in Lemma 6 and Proposition 3 is essential. Consider the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\psi^2) \quad (19)$$

It has symmetry group  $\mathbb{R} \times SO(3)$ , with  $\partial_t$  generating the commuting  $\mathbb{R}$  symmetry. Thus the element  $T \in \mathbb{R}T \oplus \mathfrak{so}(3)$  such that  $\partial_t \equiv X^T$ , is stable under the Adjoint action of  $\mathbb{R} \times SO(3)$ . The invariant cone  $\{\lambda T / \lambda \geq 0\}$  is mapped into the future cones  $C_p$  at  $p \in \mathcal{M}$  only when  $r \geq 2M$ . The group action  $\Gamma$ , though it is fully causally preserving on the region  $r \geq 2M$ , is not fully causally preserving on the whole of spacetime. Since inside the horizon all the Killing vectors become spacelike, clearly  $\bigcap_{p \in \mathcal{M}}\{A \in \mathfrak{g} / X_p^A \in C_p\} = \emptyset$ , and the Schwarzschild spacetime is not static. This example also shows that Corollary 1 is not true when the spacetime is not homogeneous. The following conditions for staticity will be more useful:

**Corollary 3** *Let  $G$  a Lie group act transitively on a manifold  $\mathcal{M}$ , and suppose that for all  $g \in G$ , the  $\nu_g$  are causally preserving diffeomorphisms of the Segal structure  $(\mathcal{M}, C_p)$ . If the Adjoint invariant cone  $C_e = \bigcap_{p \in \mathcal{M}}\{A \in \mathfrak{g} / X_p^A \in C_p\}$  has non-empty interior in  $\mathfrak{g}$ , then  $(\mathcal{M}, C_p)$  is static. Furthermore, if  $G$  is simple, then  $(\mathcal{M}, C_p)$  is static if and only if  $C_e \neq \{0\}$ . Thus if  $(\mathcal{M}, C_p)$  is observer-independent and  $G$  is simple,  $(\mathcal{M}, C_p)$  is static.*

*Proof:* We need to show that  $\text{Int}(C_e) \neq \emptyset$  implies (ii) of Proposition 3, since (i) follows immediately. Let  $A \in \text{Int}(C_e)$ , so that there exists an open neighborhood of the origin  $\mathcal{N}$  such that  $A + \mathcal{N} \subset C_e$ . Let  $p \in \mathcal{M}$ . By definition of  $C_e$ ,  $X_p^A + \mu_p \ast (e)(\mathcal{N}) \subset C_p$ . Since the action is transitive,  $\mu_p \ast (e) : \mathfrak{g} \rightarrow T_p \mathcal{M}$  is onto. Let  $\mathfrak{m}_p$  such that  $\mathfrak{g} = \text{Ker}(\mu_p \ast (e)) \oplus \mathfrak{m}_p$ , so that  $\mu_p \ast (e) : \mathfrak{m}_p \rightarrow T_p \mathcal{M}$  is an isomorphism and hence an open mapping for the induced topology on  $\mathfrak{m}_p$ . It follows that  $\mu_p \ast (e)(\mathcal{N} \cap \mathfrak{m}_p)$  is an open in  $T_p \mathcal{M}$ , so that  $X_p^A + \mu_p \ast (e)(\mathcal{N} \cap \mathfrak{m}_p)$  is a neighborhood of  $X_p^A$  contained in  $C_p$ , hence  $X_p^A \in \text{Int}(C_p)$ .

When  $G$  is simple,  $C_e = \{0\}$  or  $\text{Int}(C_e) \neq \emptyset$ , thus staticity of  $(\mathcal{M}, C_p)$  is equivalent to  $C_e \neq \{0\}$ . If  $(\mathcal{M}, C_p)$  is observer-independent,  $C_e$  in Proposition 2 is such that  $\{0\} \neq C_e \subset \bigcap_{p \in \mathcal{M}}\{A \in \mathfrak{g} / X_p^A \in C_p\}$ .  $\square$

**Corollary 4** *Let  $(\mathcal{M}, h)$  a time-oriented Lorentzian spacetime with Segal structure  $(\mathcal{M}, C_p)$ , and let  $G$  a Lie group act on  $\mathcal{M}$  as in Lemma 2. Suppose that for each  $p \in \mathcal{M}$ , the  $G$ -orbit of  $p$  has dimension 2 at least. If the Adjoint-invariant cone  $C_e \equiv \bigcap_{p \in \mathcal{M}}\{A \in \mathfrak{g} / X_p^A \in C_p\}$  has non-empty interior, then  $(\mathcal{M}, C_p)$  is static.*

*Proof:* We follow the previous proof, and if  $A \in \text{Int}(C_e)$ , there is a “neighborhood” of  $X_p^A$  which is at least two-dimensional and sits in  $C_p$ , so that  $X_p^A$  cannot be light-like.  $\square$

We shall see on an example in the next subsection that we can have  $A \in C_e \setminus \text{Int}(C_e)$  and yet  $X^A$  static.

We now relate staticity to observer-independence of the causal structure. For a Segal structure  $(\mathcal{M}, C_p)$  with causally preserving diffeomorphisms  $\nu_g$ , we say a cone  $C_p \subset T_p \mathcal{M}$  is *homogeneous* if the action of the  $\nu_h \ast (p)$  for  $h \in H_p$  on  $T_p \mathcal{M}$  is transitive

on the rays of the interior of  $C_p$ . In a spacetime  $(\mathcal{M}, h)$ , the  $\nu_h$  generate a subgroup of the restricted Lorentz group  $SO(1, n-1)^+$ .

**Lemma 7** *Let  $(\mathcal{M}, C_p)$  a Segal structure, and let  $G$  act on  $\mathcal{M}$  such that for all  $g \in G$ , the  $\nu_g$  are causally preserving diffeomorphisms of  $(\mathcal{M}, C_p)$ . Suppose  $(\mathcal{M}, C_p)$  is static, so that  $C_e = \bigcap_{p \in \mathcal{M}} \{A \in \mathfrak{g} / X_p^A \in C_p\}$  is a non-trivial  $Ad_G$ -invariant closed convex cone in  $\mathfrak{g}$ . If there exists  $p \in \mathcal{M}$  such that  $C_p$  is a homogeneous cone, and  $\mu_p * (e)C_e$  is closed in  $T_p \mathcal{M}$ , then  $(\mathcal{M}, C_p)$  is observer-independent.*

*Proof:* Let  $X^A$  a static Killing field so that  $A \in C_e$ , and let  $V_p \in Int(C_p)$ . By hypothesis there exists  $h \in H_p$  and  $\lambda > 0$  such that  $V_p = \nu_h * (p)(\lambda X^A) = X_p^{Ad_h(\lambda A)}$ , using (9). The Killing field  $X^{Ad_h(\lambda A)}$  is an extension of  $V_p$  which is causal since  $Ad_h A \in C_e$ . Thus  $Int(C_p) \subset \mu_p * (e)C_e \subset C_p$ , so that  $\mu_p * (e)C_e = C_p$  if  $\mu_p * (e)C_e$  is closed. The action  $\Gamma$  is fully causally preserving by Proposition 3, and  $\mathcal{M}$  is necessarily homogeneous, so we can apply Proposition 2.  $\square$

For example, the staticity of Anti-de-Sitter space and Minkowski space implies the observer-independence of their causal structures.

### 5.3 The inequivalent times of $AdS_2$

We now give a detailed application of the results so far. Two-dimensional Anti-de-Sitter space,  $AdS_2$ , appears (with an  $S^2$  factor), as the near-horizon geometry of the extremal Reissner-Nordström black-hole, and seems to encode some of the properties of the latter [20]. Moreover, type 0A string theory on  $AdS_2$  is conjectured to be related to matrix models [21]. The inequivalence of certain quantum field theory vacua in Anti-de-Sitter spaces often plays a major role in both the semi-classical study of black-hole physics and the  $AdS/CFT$  correspondence. We will see that the mathematical tools introduced so far and the notions of change of inertial frame and change of observers of Sec.4 have interesting physical applications, especially to distinguish the global time, the Poincaré time and the Schwarzschild time of  $AdS_2$ . Our remarks generalise quite simply to  $AdS_n$ .

The line element of (the universal cover of)  $AdS_2$  in global coordinates reads:

$$ds^2 = \frac{R^2}{\sin^2 \sigma} (-d\tau^2 + d\sigma^2) \quad (20)$$

where  $0 \leq \sigma \leq \pi$ ,  $\tau \in \mathbb{R}$  and we identify  $\tau \equiv \tau + 2\pi$  to get  $AdS_2$ . We take the curvature radius to be  $R = 1$ , and denote the metric by  $h$  as usual. Its Killing vectors (with  $\partial_\tau \equiv \partial/\partial\tau$  and  $\partial_\sigma \equiv \partial/\partial\sigma$ ) are given by:

$$\begin{aligned} X^T &= \frac{1}{\sqrt{2}} \partial_\tau, & X^Y &= \frac{1}{\sqrt{2}} (\cos \tau \cos \sigma \partial_\tau - \sin \tau \sin \sigma \partial_\sigma) \\ X^Z &= -\frac{1}{\sqrt{2}} (\sin \tau \cos \sigma \partial_\tau + \cos \tau \sin \sigma \partial_\sigma) \end{aligned} \quad (21)$$

and satisfy the commutation relations:

$$[X^T, X^Y] = \frac{1}{\sqrt{2}} X^Z, \quad [X^T, X^Z] = -\frac{1}{\sqrt{2}} X^Y, \quad [X^Y, X^Z] = -\frac{1}{\sqrt{2}} X^T \quad (22)$$

Their flow defines an effective action on  $AdS_2$  which in turn induces the following Lie-algebra anti-homomorphism:

$$\phi : A \equiv tT + yY + zZ \mapsto tX^T + yX^Y + zX^Z \quad (23)$$

where  $T$ ,  $Y$  and  $Z$  span an  $\mathfrak{sl}(2, \mathbb{R})$  Lie-algebra with commutation relations given by the opposite of (22), and  $(t, y, z)$  are the corresponding coordinates of an element  $A \in \mathfrak{sl}(2, \mathbb{R})$ . Let  $(AdS_2, C_p)$  the Segal structure stemming from the metric (20), where  $\partial_\tau$  is defined to be future-directed. By Lemma 2, the connected Lie group  $SL(2, \mathbb{R})$  acts as causally preserving diffeomorphisms of  $(AdS_2, C_p)$ .

Now  $SL(2, \mathbb{R}) \simeq Sp(2, \mathbb{R})$  is in Table 1 and thus admits bi-invariant cones. One such cone is given by its Killing form  $B : (A, B) \mapsto \text{Tr}(ad_A \circ ad_B)$ , where  $ad_A(C) \equiv [A, C]$  is the adjoint representation of the Lie algebra on itself. Indeed, we can easily check that  $B(\cdot, \cdot)$  defines a Minkowski product on  $\mathfrak{sl}(2, \mathbb{R})$ :

$$B(A, B) = -t_A t_B + y_A y_B + z_A z_B$$

Since the Killing form is invariant under the Adjoint action and the group is connected,

$$C_e \equiv \{A \in \mathfrak{sl}(2, \mathbb{R}) / t \geq 0, B(A, A) \equiv -t^2 + y^2 + z^2 \leq 0\} \quad (24)$$

defines a (pointed closed convex) cone invariant under  $Ad_{SL(2, \mathbb{R})}$ . Let  $p \in AdS_2$  the point with coordinates  $(\tau = 0, \sigma = \pi/2)$ , and let  $A = tT + yY + zZ \in C_e$ . Then  $X_p^A = \frac{1}{\sqrt{2}}(t\partial_\tau - z\partial_\sigma)$  satisfies

$$h_p(X_p^A, X_p^A) = \frac{1}{2}(-t^2 + z^2) \leq 0 \quad (25)$$

and is future directed. We clearly have  $\mu_p * (e)(C_e) = C_p$ . Corollary 2 then implies that the Segal structure  $(AdS_2, C_p)$  is observer-independent. It is static of course since  $X_p^T \in \text{Int}(C_p)$  for all  $p \in AdS_2$ . However, observer-independence of the Segal structure does not imply that all future directions are equivalent: it simply means that any future-directed vector can be extended to a global causal Killing field. Comparing (24) and (25) we see that there are some  $A \in C_e$  such that  $B(A, A) = 0$  but  $h_p(X_p^A, X_p^A) < 0$ : some ‘‘light-like’’ vectors in the group can be mapped to strictly time-like ones on spacetime.

This remark suggests to take a closer look at the ‘‘light-like’’ vectors in the Lie algebra. To do so, let us compare the global static Killing field  $\partial_\tau$  and the one which defines time in Poincaré coordinates. Let  $x \pm z \equiv \tan((\tau \pm \sigma)/2)$ , or equivalently:

$$x = \frac{\sin \tau}{\cos \tau + \cos \sigma}, \quad z = \frac{\sin \sigma}{\cos \tau + \cos \sigma}$$

These coordinates only cover  $AdS_2$  by patches, since they break down when  $\tau = \mp\sigma + (2k+1)\pi$  ( $k \in \mathbb{Z}$ ). We have  $x \in \mathbb{R}$  and  $z \geq 0$  or  $z \leq 0$  according to the patch. In each of these patches, the metric (20) reads ( $R = 1$ ):

$$ds^2 = \frac{R^2}{z^2}(-dx^2 + dz^2) \quad (26)$$

with

$$\frac{\partial}{\partial x} = (1 + \cos \tau \cos \sigma)\partial_\tau - \sin \tau \sin \sigma \partial_\sigma = \sqrt{2}(X^T + X^Y) \quad (27)$$

The Poincaré Killing field  $\partial/\partial x$  is causal and has a horizon at  $z = +\infty$ . In fact (27) is true in all the coordinate patches, and  $\partial/\partial x$  can be uniquely globally extended to the image of  $H = \sqrt{2}(T + Y) \in C_e$  under  $\phi$ . Thus it is causal everywhere, though not static. Indeed,

$$h_p(X_p^H, X_p^H) = -\frac{(\cos \tau + \cos \sigma)^2}{\sin^2 \sigma} \leq 0$$

and vanishes on the (Killing) horizons between the patches.

Proposition 1 says that the global field  $X^T$  and the Poincaré field  $X^H$  can define observables which are related by a change of inertial frame if and only if the vectors  $T$  and  $H$  are Adjoint related in  $\mathfrak{sl}(2, \mathbb{R})$ . However,  $B(T, T) < 0$  whereas  $B(H, H) = 0$ , so  $T$  and  $H$  cannot be adjoint related. Thus, the global and the Poincaré Killing fields *do not define equivalent observables*. Note also that if we restrict  $AdS_2$  to one Poincaré patch, the vector  $H$  defines a static Killing field  $X^H$ , though it is in the boundary of the cone  $C_e$ .

Moreover, some “space-like” vectors in the Lie algebra can define Killing fields which become causal in certain regions. These in fact are related to what is called in [22] the Schwarzschild time of  $AdS_2$ : the Schwarzschild-type coordinates of  $AdS_2$  are those in which the preferred causal Killing direction corresponds to the preferred time direction at spatial infinity in the near-extremal Reissner-Nordstrom black hole –of which  $AdS_2 \times S^2$  is the near horizon geometry–. The time in the near-horizon geometry is thus that which is taken to define the Boulware vacuum in the black hole space [22]. In the coordinate patch  $-\frac{\pi}{2} \pm \sigma \leq \tau \leq \frac{\pi}{2} \pm \sigma$ , let

$$\tan \frac{1}{2}(\tau \pm \sigma \mp \frac{\pi}{2}) = \mp e^{\mp q - \rho},$$

so that the  $AdS_2$  metric (20) reads:

$$ds^2 = \frac{R^2}{\sinh^2 \rho} (-dq^2 + d\rho^2)$$

where  $q \in \mathbb{R}$  is the Schwarzschild time and  $\rho > 0$  represents the inverse distance to the horizon of the black hole. The horizon of  $\partial/\partial q$  at  $\rho = +\infty$  corresponds to the black hole horizon. In fact, this horizon is precisely to  $AdS_2$  what the Rindler horizon is to Minkowski space.<sup>12</sup> One can check that the Killing field  $\partial/\partial q$  is equal to  $\sqrt{2}X^Y$  on the coordinate patch, so that it can be uniquely globally extended to that Killing field. However,  $\sqrt{2}X^Y$  is not static nor causal; its horizon corresponds to that of  $\partial/\partial q$ . Moreover,  $B(\sqrt{2}Y, \sqrt{2}Y) = 2 > 0$  and  $Y$  is “space-like” in  $\mathfrak{sl}(2, \mathbb{R})$ . It can neither be mapped to  $T$  nor to  $H$  via an Adjoint action. The Schwarzschild field  $\partial/\partial q$  defines a third non-equivalent time and corresponding energy observable; the three observables cannot be related by inertial transforms.

In quantum theory, the respective Hamiltonians of the global, the Poincaré and the Schwarzschild times, will not always have the same eigenvalues; states with positive frequencies with respect to one time variable cannot always be mapped unitarily to states with positive frequencies with respect to the other time variables. The horizons of  $X^H$  and  $X^Y$  differ in nature, since  $X^H$  remains causal globally, whereas  $X^Y$  becomes spacelike and time-like past-directed in certain regions. We expect the Hamiltonian of  $X^H$  to be better behaved than that of  $X^Y$ . We will come back to this in Sec.7. All the “strictly time-like” vectors in  $C_e$  can however be mapped to  $T$  (up to scale) via an Adjoint action, and similarly the “light-like” ones to  $H$  and the “space-like” ones to  $Z$  (up to sign): as we know, the elements of  $\mathfrak{sl}(2, \mathbb{R})$  fall into three types more often called the compact ones (like  $T$ ), the parabolic ones (like  $H$ ), and the non-compact ones (like  $Z$ ). There exists inequivalent observers in  $AdS_2$  with (local) times corresponding to each of these types.

<sup>12</sup>Rindler space appears (times  $S^2$ ) as the near-horizon limit of the Schwarzschild solution, and Rindler time corresponds to  $t$  in (19). In Minkowski space the Rindler Killing vector  $\partial/\partial t$  is a velocity boost generator, which is also “spacelike” (or non-compact) in  $\mathfrak{so}(1, 1)$ .

## 6 Horizons in the group of motions

### 6.1 Definitions

We now consider general spacetimes with Segal structures which are not necessarily observer-independent or static, and develop a novel group theoretical description of the horizon structure of spacetimes, which takes into account the notions of change of inertial frame and change of observer of Sec.4. Horizons occur in a spacetime whenever a Killing field is not static. Let  $(\mathcal{M}, h)$  a time-orientable (connected) Lorentzian manifold equipped with its Segal structure and its connected symmetry group  $G$  as in Lemma 2.

**Definition 8** *Let  $X^A$  a Killing vector field of  $(\mathcal{M}, h)$ , and suppose  $X_p^A \in \text{Int}(C_p)$  for some  $p \in \mathcal{M}$ . The horizon of an observer at  $p$  with future  $X_p^A$  is the following set:*

$$\mathcal{X}_A \equiv \{ q \in \mathcal{M} / h_q(X_q^A, X_q^A) = 0 \} \quad (28)$$

*It is empty if and only if  $X^A$  is a static Killing field.*

(We could also define the horizon of  $X^A$  for any Segal structure as the set of points  $q \in \mathcal{M}$  where  $X_q^A$  is in the topological boundary of or is extreme in  $C_q$ ). Definition 8 includes, as particular cases, Killing horizons [39] such as the  $r = 2M$  surface [40] of the Schwarzschild solution (19), cosmological horizons [40] such as those of de-Sitter space, boundaries of Ergo-regions such as in the Kerr solution [41], the Rindler horizon of Minkowski space... These features of the causal structure of spacetimes are related to physical properties such as Hawking radiation or possible energy extraction from a region of spacetime. In the physical applications, the choice of a Killing field to extend a given causal vector at a point is not arbitrary; it is in fact decisive. In Minkowski space  $\mathbb{E}^{1,3}$  with flat coordinates  $(t, x, y, z)$ , the Killing vector fields  $\partial_t$  and  $\partial_t + y\partial_x - x\partial_y$  coincide at the origin, but only  $\partial_t$  is static. The horizon of  $\partial_t + y\partial_x - x\partial_y$  is generally not thought to be relevant. However, the example of  $AdS_2$  in Sec.5.3 showed that even in a spacetime with a static or observer-independent causal structure, horizons which at first seem to appear from a bad choice of time coordinate, may in fact be given a physical interpretation –a black hole horizon in this case–. The Rindler horizon of Minkowski space is similar example (see footnote p.27). The fact that the horizon of any Killing field of  $(\mathcal{M}, h)$  should be relevant to an observer stationary with respect to this field, will be motivated in quantum theory in Sec.7.

For homogeneous spacetimes, we can lift  $\mathcal{X}_A$  to a locus in  $G$ :

**Definition 9** *Let  $(\mathcal{M}, h)$  a homogeneous Lorentzian spacetime, and let  $G$  its connected group of motions as in Lemma 2. For  $X^A$  is a Killing vector field of  $(\mathcal{M}, h)$  which is timelike future-directed at  $p$ , we define the horizon in  $G$  for an observer at  $p \in \mathcal{M}$  with future  $X_p^A$  to be the set*

$$\mathcal{H}_{A,p} \equiv \{ g \in G / h_{g,p}(X^A, X^A) = 0 \} = \mu_p^{-1}(\mathcal{X}_A) \quad (29)$$

$\mathcal{H}_{A,p}$  is also defined when the spacetime is not homogeneous; however, it can be empty when  $\mathcal{X}_A$  is not. This occurs typically in black-hole spacetimes such as (19), where the group action cannot move along the radial direction, so that the  $G$ -orbits through points outside the black-hole do not intersect its horizon. In these cases, the near-horizon geometries might be homogeneous and encode some of the horizon structure.

$\mathcal{X}_A$ , in a homogeneous spacetime, is often thought as a surface or rather a cylinder surrounding an observer at  $p \in \mathcal{M}$ . The region of spacetime inside the horizon  $\mathcal{X}_A$ , is that for which particles stationary with respect to the observer with future  $X^A$  have causal trajectories.  $\mathcal{H}_{A,p}$  on the other hand can be thought as a surface in  $G$  “surrounding” the identity element  $e$ . Definition 9 enables us to “center” at the same point  $e \in G$  all the horizons in  $\mathcal{M}$  for different observers with different future directions. As we shall see, it turns out that these different horizons in  $G$  can be elegantly compared. To do this, we first lift the spacetime metric  $h$  to a degenerate metric on  $G$ .

## 6.2 Degenerate metrics on $G$

Given  $(\mathcal{M}, h)$ , the metric tensor field  $h$  pulls back to  $G$  using for fixed  $p \in \mathcal{M}$  the map  $\mu_p : G \rightarrow \mathcal{M}$ . Define  $\tilde{h}^p \equiv \mu_p^*(h)$  for fixed  $p$  so that

$$\begin{aligned} \forall X_g, Y_g \in T_g G, \quad \tilde{h}^p_g(X_g, Y_g) &\equiv [\mu_p^*(h)](X_g, Y_g) \\ &\equiv h_{g,p}(\mu_p*(g)X_g, \mu_p*(g)Y_g) \end{aligned}$$

The Killing vector fields correspond on each  $G$ -orbit in  $\mathcal{M}$  to the images by  $\mu_p$  of the *right*-invariant fields  $\tilde{A}_g = R_g * (e)A$  on  $G$ . In contrast, the metric  $\tilde{h}^p$  on  $G$  is *left*-invariant. Indeed, for  $A, B \in \mathfrak{g}$ , using  $\nu_g \circ \mu_p = \mu_p \circ L_g$  and equation (4), we get

$$\begin{aligned} \tilde{h}^p_g(L_g * (e)A, L_g * (e)B) &\equiv h_{g,p}(\mu_p*(g)L_g * (e)A, \mu_p*(g)L_g * (e)B) \\ &\equiv h_{g,p}(\nu_g * (p)\mu_p * (e)A, \nu_g * (p)\mu_p * (e)B) \\ &= h_{g,p}(\nu_g * (p)X_p^A, \nu_g * (p)X_p^B) = h_p(X^A, X^B) \\ &\equiv \tilde{h}^p_e(A, B) \end{aligned}$$

and hence  $L_g * \tilde{h}^p = \tilde{h}^p$ . Thus  $\tilde{h}^p$  is simply determined by its behavior on  $\mathfrak{g} \simeq T_e G$ . For convenience we simply denote by  $\langle \cdot, \cdot \rangle_p$  the corresponding inner product on  $\mathfrak{g}$  so that

$$\forall A, B \in \mathfrak{g}, \quad \langle A, B \rangle_p \equiv h_p(X^A, X^B). \quad (30)$$

Since  $h$  is a non-degenerate Lorentzian metric, the kernel of  $\langle \cdot, \cdot \rangle_p$  defines an (null) isotropic subspace of  $h$  in  $T_p \mathcal{M}$  which has dimension 1 at most. The map  $\mu_p * (e)$  has kernel  $\mathfrak{h}_p$  the Lie algebra of the stabiliser subgroup  $H_p$ , thus the Kernel of  $\langle \cdot, \cdot \rangle_p$  has dimension  $(\dim \mathfrak{h}_p) + 1$  at most. When  $\mathcal{M}$  is homogeneous, the  $\mu_p * (e)$  are surjective and  $\langle \cdot, \cdot \rangle_p$  has kernel  $\mathfrak{h}_p$  precisely, so that if  $G$  acts simply transitively (i.e.  $\mathfrak{h}_p = \{0\}$ ),  $\langle \cdot, \cdot \rangle_p$  defines a left-invariant Lorentzian metric on  $G$ .

## 6.3 Horizons in $G$ and their properties

Using (4) and (9), the inner products  $\langle \cdot, \cdot \rangle_p$  on  $\mathfrak{g}$ , for different  $p \in \mathcal{M}$ , satisfy:

$$\forall A, B \in \mathfrak{g}, \quad \langle A, B \rangle_{g,p} = h_{g,p}(X^A, X^B) = \langle Ad_{g^{-1}}A, Ad_{g^{-1}}B \rangle_p \quad (31)$$

This equation gives the metric product of Killing vector fields of  $(\mathcal{M}, h)$  as one moves along the  $G$ -orbit of a point  $p \in \mathcal{M}$ . The non-commutativity in  $G$ , encoded in the Adjoint action, becomes essential. Equation (29) then reads:

$$\mathcal{H}_{A,p} \equiv \{g \in G / \langle Ad_{g^{-1}}A, Ad_{g^{-1}}A \rangle_p = 0\}, \quad (32)$$

and  $\mathcal{H}_{A,p}$  is completely defined group theoretically. As a consequence, the identity component of the centre of  $G$ , as that of  $H_p$ , is “inside” the group horizon  $\mathcal{H}_{A,p}$ . The same is true, by continuity of the map  $g \mapsto \langle Ad_{g^{-1}}A, Ad_{g^{-1}}B \rangle_p$ , of all elements of a sufficiently small neighborhood of the identity element  $e \in G$ .

More importantly, equation (32) underlines the physical implications of the existence of (possibly degenerate) *bi-invariant* Lorenztian metrics on Lie groups. Indeed, these can then be restricted to a homogeneous space  $\mathcal{M} \simeq G/H$  on which they are non-degenerate. The corresponding Segal structure is then trivially observer-independent, static, and all the Killing fields which are causal at one point have no horizons. This is the case for Lie groups admitting bi-invariant Lorentzian metrics, so for the maximally supersymmetric backgrounds of chiral supergravity in six dimensions [27]. In these backgrounds, the Killing fields stemming from the Lie group description have no horizons, but this does not preclude extra Killing fields from having horizons. We can now easily express how horizons behave under changes of inertial frames.

**Proposition 4** *Let  $\mathcal{H}_{A,p}$  the horizon in  $G$  of an observer at  $p \in \mathcal{M}$  with future  $X_p^A$ . Then for all  $g, k \in G$  we have  $\mathcal{H}_{Ad_g A, k.p} = g\mathcal{H}_{A,p}k^{-1}$ , which implies:*

- (i)  $\mathcal{H}_{Ad_g A, g.p} = g\mathcal{H}_{A,p}g^{-1}$ , so that changes of inertial frames in  $\mathcal{M}$  induce group conjugations on the horizons in  $G$ ,
- (ii)  $\mathcal{H}_{A,k.p} = \mathcal{H}_{A,p}k^{-1}$ , so that the horizons of observers with the same future are related by group translation.

*Proof:* Using equation (31) and (32), we have

$$\begin{aligned} h \in \mathcal{H}_{Ad_g A, k.p} &\Leftrightarrow \langle Ad_{h^{-1}}Ad_g A, Ad_{h^{-1}}Ad_g A \rangle_{k.p} = 0 \\ &\Leftrightarrow \langle Ad_{k^{-1}h^{-1}g}A, Ad_{k^{-1}h^{-1}g}A \rangle_p = 0 \\ &\Leftrightarrow g^{-1}hk \in \mathcal{H}_{A,p} \Leftrightarrow h \in g\mathcal{H}_{A,p}k^{-1} \quad \square \end{aligned}$$

With the inverse exponential, we can further lift the horizons to  $\mathfrak{g}$ , and show that conjugation in (i) becomes the Adjoint action. Property (ii) for  $h \in H_p$  implies that one can map  $\mathcal{H}_{A,p}$  to  $\mathcal{M} \simeq G/H_p$ , and get a bijection between  $\mathcal{H}_{A,p}/H_p$  and  $\mathcal{X}_A$ , as expected. (i) says that any two inertially related observers in  $\mathcal{M}$  have conjugate horizons in  $G$  (so in bijection).<sup>13</sup> At a given spacetime point  $p \in \mathcal{M}$ , the inequivalent future directions classify as the projective orbits of  $\mu_p^{-1}(Int(C_p)) \equiv \{A \in \mathfrak{g}/X_p^A \in Int(C_p)\}$  under the action of  $Ad_{H_p}$ . For any  $A \in \mu_p^{-1}(Int(C_p))$  and  $h_1, h_2 \in H_p$ , we have

$$\mathcal{H}_{Ad_{h_1}A, h_2.p} = h_1\mathcal{H}_{A,p}h_2^{-1} = h_1\mathcal{H}_{A,p}$$

Thus the set of inequivalent horizons for observers at  $p \in \mathcal{M}$  is in one-to-one correspondence with the set of orbits of  $\{\mathcal{H}_{A,p}/A \in \mu_p^{-1}(Int(C_p))\}$  under left multiplication by  $H_p$ . For instance, since the Poincaré and the Schwarzschild Killing fields are not even Adjoint related in  $\mathfrak{sl}(2, \mathbb{R})$ , their respective horizons in  $AdS_2$  are neither equivalent for observers at the same point, nor equivalent for inertially related observers.

Proposition 4 should shed a light on the nature of physical properties of horizons in a spacetime. These features should be invariant under changes of inertial frames, and transform covariantly under changes of observers. When lifted to the symmetry group, they are expected to transform accordingly under group conjugation or translation. For example, the expressions for the de-Sitter temperature [42] and the Unruh temperature [43], when transcribed in this group theoretical formalism, must be invariant

<sup>13</sup>Analogously, all of our line-of-sight horizons on earth define conjugate loci in  $SO(3)$ !

under group conjugation.

Equation (32) shows how horizons can be thought of as arising merely from the composition law of the symmetry group of spacetime: the metric at one spacetime point  $p \in \mathcal{M}$  together with the Adjoint action of the group, suffice to determine completely the horizons of all observers in  $\mathcal{M}$ . This gives an elegant description of the horizon structure of de-Sitter space or that of the Poincaré Killing fields of Anti-de-Sitter space. Moreover, this definition might be helpful to study the topology of horizons.

## 7 Positive energy and bi-invariant cones

We now look at the relations between properties of infinitesimal causal structures on a manifold and the possibility to define a positive energy for states of a physical theory with spacetime interpretation. We say that a theory has positive energy whenever there exists an energy observable whose associated functional is one-side bounded. We shall see that for quantum theories, the existence of particular invariant cones in the symmetry group yields a necessary and sufficient condition.

### 7.1 Causal Killing fields and observer-independent classical observables

We consider a time-orientable Lorentzian spacetime  $(\mathcal{M}, h)$ , together with its group of Killing symmetries as in Lemma 2. We already mentioned that Killing symmetries of  $(\mathcal{M}, h)$  yield conserved quantities along the geodesics: given a geodesic  $\sigma \mapsto (x_\mu)(\sigma)$  and a Killing vector field  $K^\mu$ , the quantity  $K^\mu D x_\mu(\sigma)/D\sigma$  does not depend on  $\sigma$ . It is usually called energy, momentum, or angular momentum, according to whether  $K^\mu$  generates a time translation, a space translation or a rotation in  $\mathcal{M}$ . Letting  $K^\mu \equiv X^A$ , the functional

$$V_p \mapsto h_p(X_p^A, V_p) \quad (33)$$

is positive on  $C_p$  for all  $p \in M$  if and only if  $X^A$  is causal. It can be composed with  $\Psi_p$  in diagram (2) to define an observable-functional on the states of a general theory. However, any causal vector field  $T$  on  $(\mathcal{M}, h)$  similarly defines a negative functional. Why should these be discarded? In fact in general relativity there is no reason to privilege even a particular *local* future from another one. The major discrepancy between classical and quantum theories stems from the difference between local and global descriptions. A theory of causal curves does not need a global future or global observables: time evolution has a sense locally. As soon as one goes to a global picture, one tends to replace the notion of localized observer by that of reference frame or “family of observers”, and hence think that the time parameter in the Schrödinger-type equation should correspond to a global future in spacetime. This is misleading. The time parameter of the unitary evolution need not be a global future on spacetime in the sense of causal vector fields; it is attached to a particular observer. The specificity of Killing fields, as we shall show in the next section, is precisely that they all allow for a *Schrödinger-type evolution* in the Hilbert space. In that sense, whether they are (globally) causal or not, they are preferred mathematical tools to define time evolution in quantum theory, for particular observers.

Now the fact that the Killing fields should be causal is another issue. Well-defined time evolution does not imply positive energy. If one accepts that, in our universe, future-directedness can be an observer-dependent concept –by which we mean that

the Segal structure of our spacetime need not be observer-independent–, then the unboundedness of the energy simply comes with it. Such an assumption is not absurd. It is evident that given a particle with velocity  $V_p$  at  $p \in \mathcal{M}$ , two observers at  $p \in \mathcal{M}$  boosted with respect to each other measure different values for (33). Now whatever the spacetime, their values have the same sign of course. For observers at different points, these values might have different signs: this just means that mathematically we are not able to define a common Killing future direction, and hence define the *same* unitary evolution in the Schrödinger picture for both observers. However, in the real world, an observer at  $q \neq p$  cannot measure anything at  $p$ , and thus such measurements have no meaning. If we think that an observer in de-Sitter space cannot measure a state at his antipodal point, there is no paradox in having unbounded energy functionals. It becomes a problem of interpretation.

Lemma 6 gives the necessary and sufficient conditions for a spacetime  $(\mathcal{M}, h)$  to admit a bounded energy functional  $V_p \mapsto h_p(X_p^A, V_p)$ , and the existence of an Adjoint-invariant cone in the symmetry group is required.

Furthermore, equation (4) implies the values of all observables (33) on a given state represented by  $V_p \in C_p$ , remain the same under changes of inertial frames: both the state and the observables are mapped with  $\nu_g * (p)$  to physically equivalent ones. Of course this is not true for changes of frames induced by conformal transformations of  $(\mathcal{M}, h)$ . On the other hand requiring that these functionals be invariant under changes of observers highly restrains the possibilities. By this we mean that the outcome on a given state represented by  $V_p \in C_p$  does not vary under the changes of observers of Sec.4.3: the Killing field  $X^A$  must be such that for all  $p \in \mathcal{M}$  and  $V_p \in T_p \mathcal{M}$  fixed, we have

$$\begin{aligned} \forall g \in G, \quad h_{g,p}(X_{g,p}^A, \nu_g * (p)V_p) &= h_p(X_p^{Ad_{g^{-1}} A}, V_p) \\ &= h_p(X_p^A, V_p). \end{aligned} \quad (34)$$

**Lemma 8** *Let  $G$  a maximal (connected) Lie group act effectively on an  $n$ -dimensional spacetime  $(\mathcal{M}, h)$  as its Killing motions. Then the observable  $V_p \mapsto h_p(X_p^A, V_p)$  is invariant under changes of observers if and only  $A \in \mathfrak{g}$  is invariant under the Adjoint action of  $G$ , or equivalently  $A$  is in the center of  $\mathfrak{g}$ . This implies that for all  $p \in \mathcal{M}$  such that  $X_p^A \neq 0$ ,  $H_p \subset SO(n-1) \subset SO(1, n-1)^+$ , so that there are no “boost symmetries” at this point.*

*Proof:* Condition (34) is satisfied trivially when  $A \in \mathfrak{g}$  is  $Ad_G$ -invariant. Since  $h_p$  is non degenerate,

$$\{X \in T_p \mathcal{M} / \forall V_p \in T_p \mathcal{M}, h_p(X_p^A + X, V_p) = h_p(X_p^A, V_p)\} = \{0\}.$$

As a consequence, for all  $p \in \mathcal{M}$  and  $g \in G$ ,  $Ad_{g^{-1}} A - A \in \text{Ker}(\mu_p * (e))$ . But  $\bigcap_{p \in \mathcal{M}} \text{Ker}(\mu_p * (e)) = \{0\}$  for an effective action, so that  $A \in \mathfrak{g}$  is  $Ad_G$ -invariant.

Let  $X^A$  such a Killing field, and suppose  $X_p^A \neq 0$  and  $X_p^A \notin C_p$ . Let  $V_p \in \text{Int}(C_p)$ , so that there exists  $\lambda > 0$  such that  $V_p \pm \lambda X_p^A \in \text{Int}(C_p)$ . If  $H_p$ , or rather the linear isotropy group  $\{\nu_h * (p) / h \in H_p\}$ , contains a boost symmetry, then there exists  $h \in H_p$  such that  $\nu_h * (p)V_p$  is arbitrarily close to the boundary of  $C_p$ . (The boosts of  $SO(1, n-1)^+$  take points “up” the hyperbolae in  $C_p$ , thus arbitrarily close to the boundary.) Then  $h \in H_p$  can be chosen such that either of  $\nu_h * (p)(V_p \pm \lambda X_p^A) = \nu_h * (p)V_p \pm \lambda X_p^A$  is not in  $C_p$ , which is a contradiction. If  $X_p^A \in C_p$ , take  $-X_p^A$ .  $\square$

For example, as we noticed earlier,  $\partial_t$  in the Schwarzschild metric (19) –or rather its antecedent by  $\phi_-$ – is invariant under the Adjoint action of the symmetry group  $\mathbb{R} \times SO(3)$ . The corresponding energy functional, though it is not bounded on the whole black-hole spacetime, is invariant under changes of observers (and changes of inertial frames of course) which relate any two different points with the same radial coordinate. This is not true of  $\partial_\phi$ , so that angular momentum is not observer independent.

Of course this notion of invariance under changes of observers is itself invariant under changes of inertial frames. Spacetimes with simple groups of motions  $G$  (with  $\dim G \geq 2$ ) cannot admit such Killing fields  $X^A$ , because  $A$  would span an ideal in  $\mathfrak{g}$ . This excludes for example Anti-de-Sitter and de-Sitter spaces, all Bianchi models with simple maximal symmetry groups... The “no-boost” condition also excludes many spacetimes, the simplest being Minkowski space  $\mathbb{E}^{1,n-1} = E(1, n-1)/SO(1, n-1)$ . Of course for physical relevance, the group of motions to consider is that induced by the symmetries of particular theory.

To go from the classical notion of positive energy considered in this section, to that defined by one-side bounded quantum operators, the structure of the universal algebra of the group of motions will be useful.

## 7.2 The universal algebra and the Dirac procedure

Remember that the universal algebra of  $\mathfrak{g}$ ,  $\mathcal{U}(\mathfrak{g})$ , is the extension of  $\mathfrak{g}$  into an associative algebra such that the commutator of two elements of  $\mathfrak{g}$  be given by their Lie bracket. Proposition 1 establishes a correspondence between  $\mathcal{U}(\mathfrak{g})$  and particular spacetime observables, which generalise those associated to Killing vector fields. These can be thought of as a set of differential operators on the spacetime  $\mathcal{M}$ , and constitute an infinite-dimensional algebra which is fundamentally non-commutative.

The Dirac procedure is usually defined between the Poisson algebra of real valued functions on phase space, and an associative algebra of operators acting on a Hilbert space. It suffers from ordering problems. In our approach, we will define a Dirac map from  $\mathcal{U}(\mathfrak{g})$  to a set of operators. There are no ordering problems at this point since  $\mathcal{U}(\mathfrak{g})$  is already a non-commutative associative algebra.

### 7.2.1 Quantum observables and changes of inertial frames

Let  $\mathfrak{g}$  the real Lie algebra of a group of symmetries  $G$  acting almost-effectively on a space  $\mathcal{M}$ .  $\hbar$  is Planck’s constant divided by  $2\pi$ , and  $i$  is  $\sqrt{-1}$ .

**Definition 10** *Let  $\pi^* : \mathfrak{g} \rightarrow \text{AHerm}(\mathcal{F})$  a representation of  $\mathfrak{g}$  into a set of anti-self-adjoint operators of a Hilbert space  $\mathcal{F}$ . The Dirac map  $\hat{\phantom{A}}$  is defined as:*

$$\begin{aligned} \hat{\phantom{A}} : \mathfrak{g} &\longrightarrow \text{Herm}(\mathcal{F}) \\ A &\longmapsto \hat{A} \equiv i\hbar\pi^*(A) \end{aligned}$$

so that for all  $A, B \in \mathfrak{g}$  we have:

$$\hat{([A, B])} = \frac{1}{i\hbar}[\hat{A}, \hat{B}] \equiv \frac{1}{i\hbar}(\hat{A}\hat{B} - \hat{B}\hat{A}). \quad (35)$$

According to the point of view, the Dirac map is induced by a unitary representation  $\pi$  of  $G$ , or it defines through exponentiation a representation of  $G = \exp \mathfrak{g}$ . For  $A \in \mathfrak{g}$  and  $v \in \mathcal{F}$ , we have:

$$\pi*(A)v \equiv \frac{d}{dt}\pi(\exp tA)v|_{t=0} \quad (36)$$

and

$$\pi(\exp A)v = \exp(\pi*(A))v$$

There are issues relating to convergence and definition of these maps, and generally speaking, the derived representation (36) is only defined on a dense subset of  $\mathcal{F}$ , the pre-Hilbert space of vectors  $v \in \mathcal{F}$  such that  $g \in G \mapsto \pi(g)v$  is smooth [19, p.388]. We shall not dwell on these problems though. Physically, the representation space  $\mathcal{F}$  corresponds to the set of pure states of the theory.<sup>14</sup> There is a unique way [23, p.90] of extending the representation  $\pi*$  of  $\mathfrak{g}$  to a representation of  $\mathcal{U}(\mathfrak{g})$  (and a representation of the latter determines  $\pi*$  completely). One then has, for all  $A, B \in \mathcal{U}(\mathfrak{g})$ ,  $\pi*(AB) = \pi*(A)\pi*(B)$ , and

$$\widehat{AB} = \frac{1}{i\hbar} \widehat{A} \widehat{B}. \quad (37)$$

The composition law on the observables that of  $\mathcal{U}(\mathfrak{g})$  up to an important factor. We denote by  $\widehat{A} \mapsto \widehat{A}^\dagger$  the adjoint operation on suitable operators, which is defined with respect to the fixed scalar product on  $\mathcal{F}$ . Using (37), for  $A \in \mathfrak{g}$  and  $A^n \in \mathcal{U}(\mathfrak{g})$ , we have:

$$[\widehat{A^n}]^\dagger = (-1)^{n+1} \widehat{A^n}.$$

Now if for physical reasons one should be allowed to compose the quantum (Killing) observables  $\widehat{A} \in \widehat{\mathfrak{g}}$  with themselves, then we need to extend  $\pi*$  by linearity to the complexified Lie algebra  $\mathfrak{g}_{\mathbb{C}} \equiv \mathfrak{g} \oplus i\mathfrak{g}$ : indeed, as an immediate consequence of (37),

$$(\widehat{A})^n = \widehat{((i\hbar)^{n-1} A^n)},$$

which is self-adjoint for all  $n$ , contrary to  $\widehat{A^n}$ . Complex conjugation in  $\mathfrak{g}_{\mathbb{C}}$  preserves the Lie bracket, whereas the ‘dagger’ operation  $\widehat{A} \mapsto \widehat{A}^\dagger$  changes the sign of the commutator. Compatibility is recovered by defining the ‘ $*$ ’ conjugation on  $\mathfrak{g}_{\mathbb{C}}$  as  $A + iC \mapsto [A + iC]^* \equiv -A + iC$ , which extends to  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  in a way that  $(AB)^* = B^* A^*$  for all  $A, B \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ . Then  $\pi*$  defines a hermitian representation of the algebra  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  [19, p.30], and for all  $A \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ ,

$$\widehat{(A^*)} = -[\widehat{A}]^\dagger \quad (38)$$

Physically we are only interested in self-adjoint operators, i.e. those which yield real valued functionals, but the unique extension of  $\pi*$  to  $\mathcal{U}(\mathfrak{g})$  does not generically yield such operators. In the general case, (38) says it suffices to take elements  $A \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  such that  $A^* = -A$ . Call  $\text{Herm}(\mathcal{U}(\mathfrak{g}_{\mathbb{C}}))$  the set of such elements. As a real vector space,  $\text{Herm}(\mathcal{U}(\mathfrak{g}_{\mathbb{C}}))$  is in fact isomorphic to  $\mathcal{U}(\mathfrak{g})$ . Indeed, by using the Poincaré-Birkhoff-Witt theorem [35, Theorem 2.1.11], one gets:

$$\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\mathfrak{g}) \oplus i\mathcal{U}(\mathfrak{g})$$

<sup>14</sup>As mentioned before, in order to have real linear expectation value functionals, we should be considering density matrices over  $\mathcal{F}$ . However, for stationary observers, evolution will be unitary, so that pure states remain pure.

so that an element  $A = X + iY$  is such that  $A^* = -A$  if and only if  $X = -X^*$  and  $Y^* = Y$ . Then the map

$$\begin{aligned} \text{Herm}(\mathcal{U}(\mathfrak{g}_{\mathbb{C}})) &\longrightarrow \mathcal{U}(\mathfrak{g}) \\ X + iY &\longmapsto X + Y \end{aligned} \quad (39)$$

is an isomorphism since the elements of  $\mathcal{U}(\mathfrak{g})$  decompose uniquely into a sum of self-conjugate and an anti-self-conjugate elements. Thus although we have complexified  $\mathfrak{g}$  to  $\mathfrak{g}_{\mathbb{C}}$ , we still essentially have the same classical observables. Basically in  $\mathcal{U}(\mathfrak{g})$  these are polynomials of elements of  $\mathfrak{g}$  with symmetric coefficients of odd rank and anti-symmetric coefficients of even rank, like  $A + BC - CB$  for example, while in  $i\mathcal{U}(\mathfrak{g})$  the symmetricity of the coefficients according to rank is opposite, like in  $i\hbar A^2$ .

Note that if  $\pi$  were not required to be a unitary representation of  $G$ , since  $\mathcal{U}(\mathfrak{g})$  is itself isomorphic to the algebra of left-invariant differential operators on  $G$  [23, p.98], one could take any suitable space of functions on  $G$  as a set of physical states for our physical theory.

We now consider changes of inertial frames in quantum theory. A conceptual drawback of quantization procedures is that by definition one goes from classical to quantum theory, and hence for instance one has to assume that the quantum changes of frames stem from the classical ones. This is precisely the opposite to what was done in our discussion in Sec.4.1. Given this however, the principle of relativity implies that a quantum observable which vanishes identically in an inertial frame should vanish identically in all inertial frames. In fact, we will see that this is automatically satisfied, and show that the changes of inertial frames as established in Proposition 1 for classical observables, induce the expected transforms on the quantum observables. We call  $\text{End}(\mathcal{F}) \equiv \text{Herm}(\mathcal{F}) \oplus \text{AHerm}(\mathcal{F})$  the algebra of operators on  $\mathcal{F}$  which admit an adjoint.

**Proposition 5** *Let  $\mathcal{M}$  a spacetime acted upon by a group of symmetries  $G$  with Lie algebra  $\mathfrak{g}$ , and let  $\hat{\phantom{a}}$  a Dirac map of a theory with symmetry  $G$ ,  $\hat{\phantom{a}} : \mathfrak{g} \rightarrow \text{Herm}(\mathcal{F})$ . Then  $\hat{\phantom{a}}$  extends uniquely to  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ , and for all  $g \in G$ , we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) & \xrightarrow{\text{Ad}_g} & \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \\ \hat{\phantom{a}} \downarrow & & \hat{\phantom{a}} \downarrow \\ \text{End}(\mathcal{F}) & \xrightarrow{\widehat{\text{Ad}}_g} & \text{End}(\mathcal{F}) \end{array} \quad (40)$$

Specifically, for all  $A \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ ,  $B \in \mathfrak{g}$  and  $b \in \mathbb{R}$ , we have:

$$\widehat{\text{Ad}}_{\exp bB} \widehat{A} = \text{Ad}_{\exp \frac{-ib}{\hbar} \widehat{B}} \widehat{A} = e^{\frac{-ib}{\hbar} \widehat{B}} \widehat{A} e^{\frac{ib}{\hbar} \widehat{B}} \quad (41)$$

The Adjoint action of the unitary operator  $e^{\frac{-ib}{\hbar} \widehat{B}}$  on a quantum observable, represents the effect of a spacetime change of inertial frame  $\nu_{\exp bB}$  of type  $B$  and parameter  $b \in \mathbb{R}$ . Two quantum observables for different observers can be thought of as physically equivalent if and only if they are on the same  $\text{Ad}_{\pi(G)}$ - (projective)-orbit.

*Proof.* The fact that  $\widehat{\mathcal{U}(\mathfrak{g}_{\mathbb{C}})} \subset \text{End}(\mathcal{F})$  follows from (37) together with  $\widehat{\mathfrak{g}} \subset \text{Herm}(\mathcal{F})$ . Also from (37), we see that  $\text{Ker}(\hat{\phantom{a}})$  is a two-sided ideal of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ , and as such it is

stable under  $Ad_G$  [35, Prop.2.4.17]. The commutative diagram simply follows. For  $A, B \in \mathfrak{g}$  and  $b \in \mathbb{R}$ , we have [23, p.118]:

$$Ad_{\exp bB} A = e^{ad_{bB}} A \equiv A + b[B, A] + \frac{b^2}{2}[B, [B, A]] + \dots, \quad (42)$$

This remains true for  $A \in \mathfrak{g}_{\mathbb{C}}$  and also for  $A \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  (the extension is unique), and implies, using (35):

$$\begin{aligned} \widehat{Ad}_{\exp bB} \widehat{A} &\equiv \widehat{(Ad}_{\exp bB} A) \\ &= \widehat{A} + \frac{b}{i\hbar} [\widehat{B}, \widehat{A}] - \frac{b^2}{\hbar^2} [\widehat{B}, [\widehat{B}, \widehat{A}]] + \dots \\ &\equiv e^{\frac{-ib}{\hbar} ad_{\widehat{B}}} \widehat{A} = Ad_{\exp \frac{-ib}{\hbar} \widehat{B}} \widehat{A}. \end{aligned} \quad (43)$$

Since the composition of linear operators is bilinear, (as for matrix multiplication), this expression can be simplified to the more common form:

$$Ad_{\exp \frac{-ib}{\hbar} \widehat{B}} \widehat{A} = e^{\frac{-ib}{\hbar} \widehat{B}} \widehat{A} e^{\frac{ib}{\hbar} \widehat{B}}$$

The exponential of operators is defined as the usual series:

$$e^{\frac{-ib}{\hbar} \widehat{B}} \equiv \sum_{k=0}^{k=\infty} \frac{1}{k!} \left( \frac{-ib}{\hbar} \widehat{B} \right)^k$$

It is well-defined and corresponds indeed to the Lie group exponential in the group of unitary operators  $\pi(G)$ : we have  $e^{\frac{-ib}{\hbar} \widehat{B}} = \exp(b\pi^*(B)) = \pi(\exp bB)$ .

The set of self-adjoint operators  $\widehat{\mathcal{U}(\mathfrak{g}_{\mathbb{C}})} \cap \text{Herm}(\mathcal{F})$  is stable under the Adjoint action of  $\pi(G)$ . Any two observables are said to be equivalent if they can be related up to scale by a change of inertial frame.  $\square$

Proposition 5 is the quantum analogue of Proposition 1, but whereas  $\mathcal{U}(\mathfrak{g})$  is isomorphic to the algebra of spacetime (Killing) differential operators  $\mathcal{U}(\phi(\mathfrak{g}))$ ,  $\pi^*$  does not necessarily induces a faithful representation  $\mathcal{U}(\mathfrak{g})$ . In a quantum theory, each spacetime symmetry  $B \in \mathfrak{g}$  defines a one-parameter group of unitary operators on  $\mathcal{F}$ . The elements of  $i\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$  do not define observables, nor do they correspond to changes of inertial frames, since the representation of the full group  $G_{\mathbb{C}} = \exp \mathfrak{g}_{\mathbb{C}}$  is not unitary.

We saw in the previous section that the expectation values of the Killing functionals are invariant under inertial transforms. The situation here is similar: we postulate that any physical state  $|u\rangle \in \mathcal{F}$  for an observer at  $p \in \mathcal{M}$  in a given reference frame, is taken under a spacetime inertial transform  $\nu_{\exp(bB)}$  to the physical state:

$$|v\rangle = e^{\frac{-ib}{\hbar} \widehat{B}} |u\rangle, \quad (44)$$

which implies that the expectation values of the observables in a given state are invariant under inertial transform:

$$\langle v | Ad_{\exp \frac{-ib}{\hbar} \widehat{B}} \widehat{A} | v \rangle = \langle u | \widehat{A} | u \rangle \quad (45)$$

In the classical case, the effects of changes of frames for both the velocity vectors and the Killing observables follow directly from the definition of the spacetime diffeomorphisms  $\nu_g$ . In the quantum setting we were able to show the effects of  $\nu_g$  on the

observables, but need to assume (44) to deduce the invariance law (45).

We now use the discussion in Sec.4.3 to define changes of observers between equivalent observers on the same  $G$ -orbit. Let an observer at  $p \in \mathcal{M}$  “measure” the expectation value of an observable  $\hat{A}$  on a state  $|u\rangle \in \mathcal{F}$ ,  $\langle u|\hat{A}|u\rangle$ , and let an observer at  $\exp bB.p \in \mathcal{M}$  measure the same state with the same observable. From Proposition 1, an observable  $\hat{A}$  with  $A \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  already corresponds to a globally defined (complex) observable on  $\mathcal{M}$ , so that we do not need to “extend it” to  $\exp bB.p$ . In the same way that we had to assume the changes of coordinates on the velocity states from  $V_p \in C_p$  to  $\nu_g * (p)V_p \in C_{g.p}$  (covariance rules), we postulate that the state  $|u\rangle \in \mathcal{F}$  in the reference frame at  $p$  corresponds to the state  $e^{\frac{-ib}{\hbar}\hat{B}}|u\rangle$  in the reference frame at  $\exp bB.p$ . Thus the equivalent observer at  $\exp bB.p$  measures the expectation value:

$$\langle e^{\frac{-ib}{\hbar}\hat{B}}u|\hat{A}|e^{\frac{-ib}{\hbar}\hat{B}}u\rangle = \langle u|e^{\frac{ib}{\hbar}\hat{A}}e^{\frac{-ib}{\hbar}\hat{B}}|u\rangle. \quad (46)$$

Clearly if  $|u\rangle$  did not undergo a “coordinate change”, changes of observers would have no effect, which is absurd. Equivalently, if one thinks the state  $|u\rangle$  should be fixed in  $\mathcal{F}$  but the observables undergo a change of coordinates, then going from  $\exp bB.p$  back to  $p$  in  $\mathcal{M}$  send the value of the observable  $\hat{A}$  at  $\exp bB.p$  to  $e^{\frac{\pm ib}{\hbar}\hat{B}}\hat{A}e^{\frac{-ib}{\hbar}\hat{B}}$  at  $p$ . This is analogous to  $X_{g,p}^A$  being mapped to  $X_p^{Ad_{g^{-1}}A}$ . We assume that these formulae remain valid for any observable in  $\text{Herm}(\mathcal{F})$ , and similarly for the formulae on changes of inertial frames.

### 7.2.2 Time evolution and bi-invariant cones

We now consider time evolution and observer-dependence in this formalism. The spacetime  $\mathcal{M}$  is fixed, and it has a Segal structure  $(\mathcal{M}, C_p)$ . Relativistic invariance will be guaranteed throughout by the group structure. The quantum observables constructed through the Dirac map correspond to classical observables defined on all of  $\mathcal{M}$ . It is essential however to see what these mean at a particular spacetime point, in order to interpret the quantum theory for an observer at this point. The elements of  $\mathfrak{g}$  for example classically correspond to vector fields on  $\mathcal{M}$ , and their quantum analogues should be interpreted, *for a chosen observer* at  $p \in \mathcal{M}$ , according to whether they generate time-translations, rotations, etc, *at this particular spacetime point*. Although the Hilbert space  $\mathcal{F}$  is not endowed with a causal structure of itself, we will show that choosing a spacetime observer amounts to doing so, since the unitary operators  $e^{-\frac{ib}{\hbar}\hat{B}}$  generate future displacements for an observer at  $p \in \mathcal{M}$  only if  $X_p^B \in C_p$ .

Time evolution in our setting is best defined for stationary observers in  $\mathcal{M}$ : it merely corresponds to the unitary action of a particular symmetry. Consider an observer in  $\mathcal{M}$  with future  $X_p \in \text{Int}(C_p)$  at the event  $p \in \mathcal{M}$ . This observer is stationary if there exists a Killing field  $X^T$ , for some  $T \in \mathfrak{g}$ , such that  $X_p^T = X_p$ , and such that the observer moves along the integral curve of  $X^T$  through  $p$ . This is the case for observers which measure the Boulware vacuum of the Schwarzschild black hole for example, and also for geodesic observers in de Sitter space, Minkowski space... It may well be that  $X^T$  is not causal outside a region around  $p \in \mathcal{M}$ , thus we cannot always talk of a family of equivalent observers on the  $G$ -orbit of  $p$ . For an observer translated from  $p$  by an amount  $t$  along the flow of  $X^T$ , a physical state  $|u\rangle \in \mathcal{F}$  given in the frame at  $p$  corresponds, by what we called a coordinate transform, to the following:

$$|v\rangle = e^{\frac{-it}{\hbar}\hat{T}}|u\rangle \quad (47)$$

One *postulates* that this is in fact the time evolution of the state vector *as perceived* by the observer with trajectory  $t \mapsto \exp tT.p$  in  $M$ : for all  $t \in \mathbb{R}$ ,

$$|u_T(t)\rangle \equiv e^{-\frac{it}{\hbar}\hat{T}}|u\rangle \quad (48)$$

This is the Schrödinger picture for stationary observers. The subscript  $T$  in  $|u_T(t)\rangle$  underlines the fact that there is a chosen time direction  $X^T$  in  $(\mathcal{M}, h)$ . Of course  $\hat{T}$  should be interpreted as the Hamiltonian for the given observer. It does not vary with the observer's time precisely because he is stationary. Equation (48) implies the Schrödinger-type equation:

$$i\hbar \frac{d}{dt} |u_T(t)\rangle = \hat{T} |u_T(t)\rangle \quad (49)$$

Mathematically, this holds in fact for any Killing vector field  $X^B$  in  $(\mathcal{M}, h)$ : letting  $|u_B(b)\rangle \equiv e^{-\frac{ib}{\hbar}\hat{B}}|u\rangle$  for any  $B \in \mathfrak{g}$  and  $b \in \mathbb{R}$ , we have:

$$i\hbar \frac{d}{db} |u_B(b)\rangle = \hat{B} |u_B(b)\rangle \quad (50)$$

If  $X^B$  is time-like future directed at a point  $q \in \mathcal{M}$ , then a stationary observer moving along  $b \mapsto \exp bB.q$  will see the quantum states evolve according to the Schrödinger-type equation (50), as opposed to (49). To check questions of relativistic invariance for observables and the Schrödinger equation, the following formula is most useful [23, p.117]:

$$e^{-\frac{it}{\hbar}\hat{T}} e^{-\frac{ib}{\hbar}\hat{B}} e^{\frac{+it}{\hbar}\hat{T}} = \exp \left( e^{-\frac{it}{\hbar}\hat{T}} (-ib/\hbar)\hat{B} e^{\frac{+it}{\hbar}\hat{T}} \right).$$

Basically, properties of the Adjoint action of the unitary group  $\pi(G)$  guarantee that the dynamical laws given by (48) and (49) transform accordingly under changes of inertial frames. Details can be found in [44], where the quantum change of frame formula (41) is postulated without referring to symmetries of a spacetime.

The definition of time evolution in (48) suggests the Hilbert space  $\mathcal{F}$  should be equipped with a causal structure depending on the choice of a particular observer. For each fixed spacetime point  $p \in \mathcal{M}$ ,  $\mathcal{F}$  is endowed with a cone of infinitesimal operators which determine the possible unitary time evolutions for stationary observers at that point. Indeed, recalling that  $C_e^p \equiv \{B \in \mathfrak{g}/X_p^B \in C_p\}$ ,

$$\widehat{C}_e^p = \{\hat{B} \in \text{Herm}(\mathcal{F}) / B \in \mathfrak{g}, X_p^B \in C_p\} \quad (51)$$

defines a cone of observables. Physically,  $\widehat{C}_e^p$  is the set of possible Hamiltonians for stationary observers at  $p \in \mathcal{M}$ . If the symmetry group  $G$  induces causally preserving diffeomorphisms of the Segal structure  $(\mathcal{M}, C_p)$ , we have:

$$Ad_{\pi(g)}(\widehat{C}_e^p) = \widehat{C}_e^{g \cdot p} \quad (52)$$

As a consequence, if  $\hat{H}$  is a Hamiltonian for a stationary observer at  $p \in \mathcal{M}$ , then  $e^{-\frac{ib}{\hbar}\hat{B}}\hat{H}e^{\frac{+ib}{\hbar}\hat{B}}$  is a Hamiltonian for a stationary observer at  $\exp bB.p$ .

**Lemma 9** *Let  $\mathcal{M}$  a spacetime with Segal structure  $(\mathcal{M}, C_p)$ , and let  $G$  act on  $\mathcal{M}$ . Let  $\widehat{\gamma} : \mathfrak{g} \rightarrow \text{Herm}(\mathcal{F})$  the Dirac map of a theory with symmetry group  $G$ . Then the Hilbert space  $\mathcal{F}$  admits a global time-parameter valid for particular observers at each spacetime point if and only if  $(\mathcal{M}, C_p)$  is static.*

*Proof:* This is equivalent to the existence of a self-adjoint operator  $\hat{A} \in \widehat{\mathfrak{g}}$  such that for all  $p \in \mathcal{M}$ ,  $X_p^{\hat{A}} \in \text{Int}(C_p)$ .  $\square$

This is true whatever the state space  $\mathcal{F}$ . It really suggests that the notion of observer-independence of future-directedness we introduced with Killing fields, is meaningful: given any two points in  $\mathcal{M}$ , two observers at these points stationary with respect to the same static Killing field, see states of the theory evolve under the same Hamiltonian. This is not possible in non-static spacetimes. Here again, when the  $\nu_g$  are causally preserving diffeomorphisms of  $(\mathcal{M}, C_p)$ ,

$$\widehat{C}_e \equiv \bigcap_{p \in M} \widehat{C}_e^p \quad (53)$$

defines a sub-cone of the cone of self-adjoint operators on  $\mathcal{F}$ , which is invariant under the Adjoint action of the finite-dimensional unitary group  $\pi(G)$ . It defines a set of “globally valid” Hamiltonians. Lemma 6 and the previous Lemma imply that such Hamiltonians cannot exist if  $G$  does not have an Adjoint invariant cone.

We now determine the conserved quantities and observer-independent observables. It is natural to go to the Heisenberg picture. For a stationary observer moving along the curve  $t \mapsto \exp tT.p$ , the observables  $\hat{A}$  evolve according to the  $\hat{A}_T(t) \equiv e^{\frac{i\hbar}{\hbar}\hat{T}}\hat{A}e^{-\frac{i\hbar}{\hbar}\hat{T}}$ , and the quantum states  $|u\rangle \in \mathcal{F}$  are fixed in this picture.  $\hat{A}$  is the observable in the Schrödinger picture, and it can have explicit time dependence. Using diagram (40) and the fact that  $\widehat{T}_T(t) = \widehat{T}$ , we get

$$\begin{aligned} \frac{d}{dt}\hat{A}_T(t) &= \frac{1}{i\hbar}[\hat{A}_T(t), \widehat{T}] + e^{\frac{i\hbar}{\hbar}\widehat{T}} \frac{\partial \hat{A}}{\partial t} e^{-\frac{i\hbar}{\hbar}\widehat{T}} \\ &= \frac{1}{i\hbar}e^{\frac{i\hbar}{\hbar}\widehat{T}} \left( [\hat{A}, \widehat{T}] + i\hbar \frac{\partial \hat{A}}{\partial t} \right) e^{-\frac{i\hbar}{\hbar}\widehat{T}} \\ &= \widehat{\left( \text{Ad}_{\exp -tT} \left( [A, T] + i\hbar \frac{\partial A}{\partial t} \right) \right)} \end{aligned} \quad (54)$$

We have used the linearity and smoothness of  $\widehat{\phantom{A}}$  to write  $\partial \hat{A} / \partial t$  as  $\widehat{(\partial A / \partial t)}$ . The first line remains valid for any observable in  $\text{Herm}(\mathcal{F})$ . If  $\partial \hat{A} / \partial t = 0$ ,  $\hat{A}_T(t)$  defines a constant of motion for the observer with Hamiltonian  $\widehat{T}$  if and only if  $[\hat{A}, \widehat{T}] = 0$ . The elements of

$$z(T) \equiv \{ A \in \mathcal{U}(\mathfrak{g}_C) / A^* = -A, [A, T] = 0 \}$$

satisfy these conditions. The fact that these depend upon the choice of future direction  $T$  is not surprising. For example, in Minkowski space, say there is a physical system with constant angular momentum in the  $(x, y)$  plane for an observer with future  $\partial / \partial t$ . The observable associated to  $L_z = x\partial / \partial y - y\partial / \partial x$  will satisfy  $[\widehat{T}, \widehat{L}_z] = 0$ . For a different observer with future  $\partial / \partial t + (1/2)\partial / \partial x$ , we have  $[\widehat{T} + (1/2)\widehat{X}, \widehat{L}_z] = (1/2)\widehat{Y}$  which may be non-zero. Then the observable  $\widehat{L}_z$  (and its expectation value) is not conserved for this observer, but this is expected since he moves along the  $x$ -direction.

**Definition 11** A quantum observable  $\hat{A} \in \text{Herm}(\mathcal{F})$  is called observer-independent if, for any spacetime inertial transform  $\nu_{\exp bB}$  and any physical state  $|u\rangle \in \mathcal{F}$ , we have:

$$\langle u | e^{-\frac{i\hbar}{\hbar}\widehat{B}} \hat{A} e^{\frac{i\hbar}{\hbar}\widehat{B}} | u \rangle = \langle u | \hat{A} | u \rangle \quad (55)$$

In theories where the vacuum state  $|0\rangle$  is defined to be invariant under inertial transforms, i.e.  $|0\rangle = e^{-\frac{ib}{\hbar}\hat{B}}|0\rangle$ , the vacuum expectation values of all the observables are invariant under change of observer. The Casimirs of a Lie algebra are defined as the elements of the center of its universal algebra.

**Lemma 10** *Let  $\mathcal{M}$  a spacetime and  $G$  a Lie group acting on  $\mathcal{M}$ . Let  $\hat{\cdot}$  the Dirac map of a theory with symmetry  $G$ . The observer-independent observables are the elements in*

$$\{\hat{A} \in \text{Herm}(\mathcal{F}) / \forall \hat{B} \in \hat{\mathfrak{g}}, [\hat{A}, \hat{B}] = 0\}, \quad (56)$$

*so that the anti-self-conjugate Casimirs of the Lie algebra  $\mathfrak{g}_{\mathbb{C}}$  define such observables. The latter are in one-to-one correspondence with the Casimirs of  $\mathfrak{g}$ , and their associated expectation values are called quantum numbers.*

*Proof.* Let  $\hat{A} \in \text{Herm}(\mathcal{F})$ , and suppose that for any  $b\hat{B} \in \hat{\mathfrak{g}}$ , (55) is satisfied. Since the scalar product on  $\mathcal{F}$  is non-degenerate and  $\hat{A}$  is self-adjoint, this is equivalent to  $\hat{A} = e^{-\frac{ib}{\hbar}\hat{B}}\hat{A}e^{\frac{ib}{\hbar}\hat{B}}$ . Using (54) –replacing  $t\hat{T}$  by  $b\hat{B}$ – it is equivalent to  $[\hat{A}, \hat{B}] = 0$ . This is satisfied when  $A$  is a Casimir of  $\mathfrak{g}_{\mathbb{C}}$  and  $A^* = -A$ .

One can show easily that both the self-conjugate and anti-self-conjugate part of a Casimir must be a Casimir. Since this decomposition is unique, the map (39) shows that the anti-self-conjugate part of the center of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  is isomorphic, as a vector space, to the center of  $\mathcal{U}(\mathfrak{g})$ .  $\square$

Lemma 10 is the quantum version of Lemma 8 which could be generalised any classical observables in  $\mathcal{U}(\mathfrak{g})$  or  $\text{Herm}(\mathcal{U}(\mathfrak{g}_{\mathbb{C}}))$ . For example, all simple Lie groups admit a quadradic Casimir constructed from the Killing form [45, chapter 14], whereas they do not possess non-trivial Adjoint-invariant vectors.

It is a well-known fact in that the Casimirs help to classify the irreducible representations of Lie algebras, and that they are interpreted as particular quantum numbers of a physical system with symmetry group  $G$ . The language of observer-dependence is meaningful: typically, angular momentum about an axis can be observer-dependent, whereas spin is not. Using equation (54), the most important physical consequence of observer-independence, is that the Casimirs yield constants of motion for *all* inequivalent stationary observers of spacetime, *whatever their respective Hamiltonians*.

### 7.2.3 Inequivalent Hamiltonians

We have not mentioned physical issues relating to the existence of a set of normalized eigen-vectors for the quantum observables. Finding basis of eigen-vectors of particular observables, especially the Hamiltonian, is an essential step towards understanding the physics. Proposition 5 shows that Hamiltonians related by a change of inertial frame (41) trivially share the same spectrum and their eigen-states are related by a unitary transform. However, physically inequivalent Hamiltonians will generically have different properties, especially regarding normalizability of the eigen-states, and boundedness of the spectrum. Stationary observers do not generically have equivalent futures, even for geodesic observers. This does not contradict the principle of relativity of course, which holds for *observers related by an inertial transform*. Consider for example a Rindler observer in Minkowski space moving on the time-like integral curve of the Killing field  $z\partial_t + t\partial_z$  through  $(t = 0, z = 1/a)$ . The Minkowski vacuum  $|0\rangle$  is defined with respect to  $\partial_t$  which defines a “global time” on the Hilbert space, as in Lemma 9. These two Killing fields are not inertially related since the former has

a horizon while the latter does not. If the Dirac map is injective, their associated Hamiltonians cannot be Adjoint related and thus are physically inequivalent. Indeed, the Rindler observer, who moves with uniform acceleration  $a$ , perceives  $|0\rangle$  not as the lowest energy eigen-state of his Hamiltonian, but as a thermal bath at temperature  $a/2\pi$  [43]. If the observers were equivalent, the vacuum  $|0\rangle$ , since it is invariant the unitary action of  $\pi(G)$ , would be an energy eigen-state for the Rindler observer, which it is not.

Our approach does not explain how to obtain this particular result, but gives a general frame to study and understand the reasons for observer-dependence in theories with symmetries acting on a spacetime:

**Proposition 6** *Let  $(\mathcal{M}, h)$  a time-orientable spacetime,  $(\mathcal{M}, C_p)$  its associated Segal structure, and  $G$  its connected group of Killing motions. Consider a quantum theory on  $(\mathcal{M}, h)$  with symmetry group  $G$ , and call  $\pi$  the associated representation of  $G$ . Suppose that the derived representation of  $\mathfrak{g}$  is faithful, and call  $H_p$  the stabiliser subgroup of  $p \in \mathcal{M}$ . Then the set of physically inequivalent Hamiltonians for stationary observers at  $p \in \mathcal{M}$ , is in one-to-one correspondence with the projective orbits of the cone*

$$C_e^p \equiv \{A \in \mathfrak{g} / X_p^A \in C_p\} \quad (57)$$

under the Adjoint action of  $H_p$  in  $\mathfrak{g}$ . Furthermore this classification is invariant under changes of inertial frames, so all the inequivalent Hamiltonians of the quantum theory on  $(\mathcal{M}, h)$ , are obtained by taking one such classification on each  $G$ -orbit in  $\mathcal{M}$ .

*Proof:* By Lemma 2, the action of  $G$  on  $(\mathcal{M}, C_p)$  is causally preserving. The map  $A \in \mathfrak{g} \mapsto X^A$  is injective, and, because of coordinate reparametrisation, each ray in  $C_e^p$  defines a stationary observer at  $p \in \mathcal{M}$  and reciprocally. Call  $\mathbb{P}(C_e^p)$  the set of rays in  $C_e^p$ . Since  $\nu_h(p) = p$  for  $h \in H_p$  and the action is causally preserving, it follows from (13) that  $Ad_h(C_e^p) = C_e^p$  and one can define  $\mathbb{P}(C_e^p)/Ad_{H_p}$ . Clearly two elements  $A, B \in \mathbb{P}(C_e^p)/Ad_{H_p}$  define the same equivalence class if and only if there exists  $h \in H_p$  such that  $X^A = \nu_h * X^B = X^{Ad_h B}$ , and  $\mathbb{P}(C_e^p)/Ad_{H_p}$  defines the set of inequivalent stationary observers at  $p \in \mathcal{M}$ .

Now  $\widehat{C_e^p}$  as in (51) is the set of Hamiltonians for these stationary observers, and when  $\pi*$  is faithful it is isomorphic to  $C_e^p$ . It follows directly from Proposition 5 that elements of  $\mathbb{P}(C_e^p)/Ad_{H_p}$  label mutually inequivalent Hamiltonians.

The fact that it suffices to consider only one point on each  $G$ -orbit simply follows from the nature of the changes of inertial frames. Formally, since  $H_{g,p} = gH_pg^{-1}$  and  $Ad_g(C_e^p) = C_e^{g \cdot p}$ , the map:

$$Ad_g : \mathbb{P}(C_e^p)/Ad_{H_p} \longrightarrow \mathbb{P}(C_e^{g \cdot p})/Ad_{H_{g,p}}$$

defines a bijection between the equivalence classes of stationary observers at  $p$  and at  $g \cdot p$ . The corresponding Hamiltonians are related according to  $\widehat{A} \mapsto \pi(g)\widehat{A}\pi(g)^{-1}$ , so that their physical properties are the same.  $\square$

Note that this is substantially different from the classification of inequivalent time-like geodesics at a point  $p \in \mathcal{M}$ . Indeed, the latter only requires finding the orbits of the tangent vectors in  $C_p$  under the linear-isotropy group  $\nu_{H_p}$ , whereas here one must consider the Killing fields.

For a fixed point  $p \in \mathcal{M}$ ,  $C_e^p$  defines a cone in  $\mathfrak{g}$ , and thus a corresponding left-invariant Segal structure  $(G, L_g * (e)C_e^p)$ . Proposition 6 tells us that classifying the inequivalent

Hamiltonians of the theory for an observer at  $p$ , amounts to classifying one-parameter subgroups of  $G$  which are causal with respect to the Segal structure  $(G, L_g * (e)C_e^p)$ , under the action of  $Ad_{H_p}$ .

We can classify the inequivalent Hamiltonians for observers in  $AdS_2$ , using the results and notation of Sec.5.3. The global, Poincaré, and Schwarzschild Killing fields correspond respectively to timelike, null and spacelike elements in  $\mathfrak{sl}(2, \mathbb{R})$  equipped with its Killing form. In the  $H, K, D$  basis of  $\mathfrak{sl}(2, \mathbb{R})$  defined by:

$$\sqrt{2}T \equiv \frac{1}{2}(H + K), \quad \sqrt{2}Y \equiv \frac{1}{2}(H - K), \quad \sqrt{2}Z \equiv D,$$

the Killing fields read:

$$\frac{\partial}{\partial \tau} = X^{\frac{1}{2}(H+K)}, \quad \frac{\partial}{\partial x} = X^H, \quad \frac{\partial}{\partial q} = X^{\frac{1}{2}(H-K)}.$$

The property of their respective Hamiltonians vary according to the theory. In quantum field theory on  $AdS_2$ , the global and Poincaré vacua are unitarily related, but differ from the Schwarzschild vacuum [22]; this should be manifest in the lowest eigenstates of the Hamiltonians  $\hat{H} + \hat{K}$ ,  $\hat{H}$  and  $\hat{H} - \hat{K}$ , but subtleties of convergence may arise as the stationary observers for  $H$  and  $H - K$  may move on light-like or spacelike curves in  $AdS_2$ , according to the point chosen.

Let us consider Maxwell-Einstein theory on  $AdS_2 \times S^2$ , and focus on the non-relativistic motion of particles with charge equal to mass. This corresponds to the motion of certain charged particles in the infinite mass limit of the near-horizon geometry of the extremal Reissner-Nordström black hole. It was shown in [46] that the dynamics is in fact equivalent to the one-particle conformal mechanics model studied by de Alfaro, Fubini and Furlan (DFF) in [47]. In spite of the large  $R$  limit taken to obtain these dynamics, the quantum operators  $\hat{H}, \hat{K}, \hat{D}$  stemming from the phase-space functions of the DFF model define a representation of  $\mathfrak{sl}(2, \mathbb{R})$ , so that the relativistic symmetries of  $AdS_2$  correspond to conformal symmetries of this non-relativistic theory. From [47], the Hamiltonian  $\hat{H} + \hat{K}$  has an evenly spaced discrete spectrum with normalizable eigen-states, whereas  $\hat{H}$  has a continuous spectrum bounded from below, and its lowest eigen-state is not normalizable. These operators govern time evolution respectively for global stationary observers or Poincaré stationary observers. Mathematically at least, their states evolve in a *different* Hilbert space: the inequivalence of the Hamiltonians becomes all the more physically relevant.

We now apply Proposition 6 to find all the inequivalent Hamiltonians at the point  $p$  with  $(\tau = \pi/2, \sigma = \pi/2)$  in the global coordinates. Since  $AdS_2$  is homogeneous, it suffices to look at this point. We have:

$$C_e^p = \{tT + yY + zZ / t \geq 0, t^2 \geq y^2\}$$

and  $H_p$  is the one-parameter subgroup generated by  $Z$ . Its Adjoint action on  $C_e^p$  is equivalent to a Lorentz velocity boost in the  $(t, y)$  plane of  $\mathfrak{sl}(2, \mathbb{R})$ . Thus for each fixed  $z \in \mathbb{R}$ , there are three projective orbits labelled by a strictly timelike element such as  $H + K + zZ$ , a null element such as  $K + zZ$ , and a null element such as  $H + zZ$ . The Hamiltonian  $\hat{H}$  here corresponds to a stationary observer moving at the speed of light (i.e. on the horizon of  $\partial_x$ ), but at other points it corresponds to strictly-timelike observers.

### 7.3 Highest weight representations

We now explain how Adjoint invariant cones in  $\mathfrak{g}$  are related to sets of positive operators on  $\mathcal{F}$ . In Lemma 9 we established that such cones were necessary in order to define of a Hamiltonian which is globally valid for particular observers in all of  $\mathcal{M}$ . The assumptions are simply that the space of physical states of the theory carries a unitary representation of a symmetry group which acts on the spacetime, thereby defining inertial transforms. Recall that this is the case in quantum field theories, and also in string theories: although one tends to concentrate on the string world-sheet Virasoro symmetries, the target space or global symmetries, whether one works in the light-cone gauge or not, are essential. The physical spaces  $\mathcal{F}$  admit a bounded notion of energy whenever there exists a self-adjoint operator which has a (one-side) bounded spectrum and which can be thought of as generating time evolution. In the Schrödinger picture for stationary observers (when possible), we would want this operator to be the image by the Dirac map of a Killing field which is causal for a particular observer in spacetime.

Representations which admit one-side bounded operators are in fact unitary highest weight representations [19, Theorem X.3.9]. They possess rich algebraic and geometric structures, whose mathematical properties can be found in [19, Part D]. From a physical point of view, the algebraic structure essentially enables the construction of Fock spaces by acting on a highest weight vector, while the geometric structure encodes the existence of positive energy operators. It has been known for some time that the de-Sitter group  $O(1, 4)$  does not admit such a representation [13]. In fact as we see now, the same is true for all groups with ambivalent conjugacy classes.

As in the previous section, let  $G$  a Lie group,  $\pi$  a continuous unitary representation of  $G$  on a Hilbert space  $\mathcal{F}$ , and  $\pi^*$  the corresponding derived representation of  $\mathfrak{g}$  as defined in (36). Strictly speaking,  $\pi^*$  is defined on a dense subset of  $\mathcal{F}$  consisting of smooth vectors. For all  $A \in \mathfrak{g}$  and  $h \in G$ , we have:

$$\begin{aligned} \pi * (Ad_h A) &= Ad_{\pi(h)} \pi * (A) \\ &= \pi(h) \pi * (A) \pi(h^{-1}), \end{aligned} \tag{58}$$

which is similar to equation (41). If  $G$  has ambivalent conjugacy classes, then for all  $A \in \mathfrak{g}$ , there exists  $h \in G$  such that:

$$Ad_h A = -A. \tag{59}$$

Since  $i\pi * (A)$  is a self-adjoint operator, let  $|v\rangle \in \mathcal{F}$  and  $\lambda \in \mathbb{R}$  such that  $i\pi * (A)|v\rangle = \lambda|v\rangle$ . Then using (59) and (58), we get:

$$\begin{aligned} i\pi * (Ad_h A) \pi(h)|v\rangle &= -i\pi * (A) \pi(h)|v\rangle \\ &= i\pi(h) \pi * (A) \pi(h^{-1}) \pi(h)|v\rangle \\ &= \lambda i\pi(h)|v\rangle \end{aligned}$$

Thus  $\pi(h)|v\rangle$  is an eigen-vector of  $\widehat{A}$  with eigen-value  $-\lambda$ . The spectrum of the observable  $\widehat{A}$  is symmetric about 0.

Now suppose  $G$  acts on a spacetime  $\mathcal{M}$  by inducing causally preserving diffeomorphisms of a Segal structure  $(\mathcal{M}, C_p)$ . The operator  $\widehat{A}$  defines a possible Hamiltonian for an observer at  $p \in \mathcal{M}$ , if  $X_p^A \in \text{Int}(C_p)$ . For such an observer the states  $|v\rangle$  and  $\pi(h)|v\rangle$  have respective energies  $\lambda = \langle v|\widehat{A}|v\rangle$  and  $-\lambda = \langle v|\pi(h)^{-1}\widehat{A}\pi(h)|v\rangle$ . By definition of changes of observers,  $-\lambda$  is also the energy measured for the state  $|v\rangle$  by an

equivalent observer at  $h.p$  with same Hamiltonian  $\hat{A}$ . This observable corresponds to the “future”  $X_{h.p}^A$  at  $h.p$ . However, since  $\nu_h$  is a causally preserving diffeomorphism, using (9) and (59) we have:

$$\nu_h * (p) X_p^A = X_{h.p}^{Ad_h A} = -X_{h.p}^A \in C_{h.p} \quad (60)$$

so that in fact  $X_{h.p}^A \notin C_{h.p}$ . The observer at  $h.p$  with Hamiltonian  $\hat{A}$  is in fact measuring minus the energy in his frame. Of course the inertially equivalent observer on the other hand measures  $\pi(h)|v\rangle$  with  $\pi(h)\hat{A}\pi(h)^{-1}$  and gets the same expectation value  $\lambda$ . More importantly, (60) implies that the two observers measuring the opposite energy for the given state, are necessarily separated by a horizon in the spacetime. From the first observer’s point of view, we could say that whereas  $|v\rangle$  is measurable in his frame,  $\pi(h)|v\rangle$  lies behind his horizon. The question is then whether one can always construct localized states [29] for theories with symmetry groups with ambivalent conjugacy classes, or just with groups which do not admit unitary highest weight representations.

With  $\text{Herm}(\mathcal{F})$  denoting the set of not necessarily bounded self-adjoint operators on  $\mathcal{F}$ , we define:

$$\text{Herm}^+(\mathcal{F}) \equiv \{O \in \text{Herm}(\mathcal{F}) / \forall v \in \mathcal{F}, \langle v|O|v\rangle \geq 0\} \quad (61)$$

The positive operators define a (non-trivial) pointed closed convex cone in  $\text{Herm}(\mathcal{F})$ .

**Lemma 11** *Let  $G$  the symmetry group of a theory, and  $\pi$  a unitary representation of  $G$  on the Hilbert space  $\mathcal{F}$  of this theory, such that the derived representation  $\pi^*$  of  $\mathfrak{g}$  is faithful. If there exists a non-trivial positive observable  $i\pi^*(A) \in \text{Herm}^+(\mathcal{F})$  for some  $A \in \mathfrak{g}$ , then  $G$  admits a non-trivial pointed  $\text{Ad}_G$ -invariant closed convex cone.*

*Proof:*  $\text{Herm}^+(\mathcal{F})$  is invariant under conjugation by unitary transforms of  $\mathcal{F}$ , so in particular for all  $g \in G$ ,  $\pi(g)\text{Herm}^+(\mathcal{F})\pi(g^{-1}) \subset \text{Herm}^+(\mathcal{F})$ . The same is true for  $i\pi^*(\mathfrak{g})$ , so  $i\pi^*(\mathfrak{g}) \cap \text{Herm}^+(\mathcal{F})$  is an  $\text{Ad}_{\pi(G)}$ -invariant pointed convex cone. By hypothesis it is non-empty. Its closure is in  $\text{Herm}^+(\mathcal{F})$  hence it is also  $\text{Ad}_{\pi(G)}$ -invariant and pointed. The inverse image of the latter by  $\pi^*$  defines a cone in  $\mathfrak{g}$  with the necessary requirements. It is pointed when  $\pi^*$  is injective, and does not contain  $-A$ .  $\square$   
This simple property is completely analogous to Lemma 6: causal Killing fields on spacetimes or positive observables associated to symmetry generators in any quantum theory, require the existence of Adjoint invariant cones in the symmetry groups. In fact the study of one-side bounded operators –rather than positive operators–, is more subtle, and requires introducing mathematical tools which go beyond the present aim of this article.

**Theorem 4** *Let  $G$  a non-compact semisimple Lie group with maximal compact subgroup  $K$ . Then  $G$  admits unitary highest weight representations if and only if  $G/K$  is a hermitian symmetric space.*

*Proof:* This is a consequence of Theorem IX.5.13 in [19] which is more general and relates unitary highest weight representations of involutive Lie algebra to the existence of invariant generating convex sets.  $\square$

Some properties of unitary highest weight representations of these groups can be found in [48]: the 3-grading of the Lie algebras of the groups  $G$  in Table 1 p.10 implies one

can construct Fock spaces by acting with creation operators on highest weight vectors. Together with Theorem 3, Theorem 4 states the equivalence between the existence of positive energy representations for semi-simple groups and that of bi-invariant pointed Einsteinian cones in their Lie algebras. These are related to causal Killing fields on manifolds, and we get the following:

**Theorem 5** *Let  $G$  a simple non-compact Lie group act almost effectively on a space-time  $\mathcal{M}$ , and suppose that the  $\nu_g$ , for all  $g \in G$ , are causally preserving diffeomorphisms of a Segal structure  $(\mathcal{M}, C_p)$ . If  $(\mathcal{M}, C_p)$  admits a causal Killing field, then there exists a unitary highest weight representation of  $G$ , so that one can define through the Dirac map a quantum theory which has a notion of positive energy.*

*Proof:* From Lemma 6, if  $(\mathcal{M}, C_p)$  admits a causal Killing field, then  $\mathfrak{g}$  admits an  $Ad_G$ -invariant (non-trivial pointed closed convex) cone, which is Einsteinian for  $G$  simple. Theorem 3 implies that  $G/K$  is a non-compact hermitian symmetric space, and we apply Theorem 4.  $\square$

This theorem applies of course to Lorentzian spacetimes with static metrics and non-compact simple Lie groups. We see that the classical definition of energy, the existence of bi-invariant cones in the group of motion, and that of a bounded quantum energy, are inherently intertwined. Theorem 5 and Lemma 9, imply that, for spacetimes with non-compact simple symmetry groups, if one requires the existence of a global time in quantum theories, then one can also choose a unitary highest weight representation, and thus have positive energy.

Our analysis of symmetries in quantum theories relies on promoting the infinitesimal generators to self-adjoint quantum observables. The discussion in Sec.7.2 on the Dirac procedure applied to the universal algebra, is relevant to any theory which requires at some point a unitary representation of a symmetry group, whether compact or non-compact, relativistic or non-relativistic. The comments on time evolution and the causal structure of the Hilbert space rely however on the existence of stationary observers. By definition this is the case in all spacetimes with  $G$ -orbits which are non-spacelike, thus in all homogeneous spacetimes of course, and most probably in all backgrounds where string theory is tractable. A different approach regarding time-evolution must be taken in the cosmological models with spatial symmetry only. For non-stationary observers, even in homogeneous spacetimes, time-evolution will require time-dependent Hamiltonians, and the energy for such observers will not generically be conserved.

The symmetry generators in the Dirac procedure are analogous to the “fundamental quantities” in Dirac’s paper [49]. Owing to the explicit group structure in the formalism given here, the observer-dependent choice of Hamiltonian does not break relativistic invariance, and at this formal stage, there is no need to introduce coordinate charts. However, the quantum observables described in this procedure are only those which correspond to the generalised Killing classical observables of Proposition 1. Although these are particularly easy to manipulate in terms of changes of frames and observers, they cannot not exhaust all the observables, especially when the space-time is not homogeneous. More fundamentally, the universal algebra only contains differential observables, and thus does not contain the analogue of position in phase-space. Mathematically, if  $1$  denotes the unit element of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ , for any two elements  $A, B \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ , we cannot have  $AB - BA = 1$  [35, Remark 2.8.3]: the commutator of any two quantum observables which classically correspond to elements of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ ,

cannot be proportional to the identity operator on  $\mathcal{F}$ . It can be central though, but to obtain standard first quantization commutation relations, one has to extend  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ .

## 8 Conclusion

The main achievement of this article is to explain the physical relevance of bi-invariant cones in symmetry groups of spacetimes, and show how they appear naturally in most supergravity and string theories currently under investigation.

In order to interpret Adjoint invariance physically, we gave a detailed description of the way symmetries act on physical theories, insisting especially on the mathematical relations between the symmetries of a space of states, and the symmetries of a spacetime. Propositions 1 and 5 show how the universal algebra of the infinitesimal symmetries both corresponds to spacetime observables –differential operators on  $\mathcal{M}$ –, and quantum observables –self-adjoint operators on  $\mathcal{F}$ –, and that changes of inertial frames in  $\mathcal{M}$  correspond to Adjoint actions on these observables. The notions of changes of inertial frames and changes of observers were clearly distinguished.

Most importantly, we showed that Adjoint invariance is fundamentally related to invariance under changes of observers in a theory. For example, Lemmas 8 and 10 show that the Casimirs of the Lie algebra of symmetries define observer-independent observables. The key novel results however are those which relate observer-independence of future-directedness in a spacetime, to algebraic properties of the group of motions. In Sec.5, we established the equivalence between the existence of particular bi-invariant cones in the algebra, and observer-independent or static causal structures. This shows that in a spacetime, the local causal structure defined by the future cones, together with the symmetry group of the metric, suffice to determine whether the spacetime admits a causal Killing field. These results imply that Segal’s assumption of bi-invariant cones is satisfied for all symmetry groups of spacetimes which admit at least one causal Killing field. As a consequence, the theory of causal symmetric spaces [18] should have applications in modern theoretical physics.

Furthermore, infinitesimal causal structures were shown to be useful to understand time evolution in quantum theories: in Sec.7.2 we gave a formal relation between local relativistic times, and particular quantum operators interpreted as the Hamiltonians for stationary observers. Lemma 9 shows the intuitive fact that spacetime staticity is equivalent to the existence, in any quantum theory, of a Hamiltonian valid for a family of stationary observers at each spacetime point, or indeed of a global time evolution on the space of states.

The notion of changes of observers enabled us to formally classify the inequivalent Hamiltonians of a quantum theory, by simply studying the orbits of the locally causal Killing vectors at a point under the action of the linear isotropy group at this point. The technique explained in Proposition 6 relates to the classification of quotients of spacetimes admitting symmetries. Over the past years, this has been widely studied in the context of Kaluza-Klein and orbifold reductions (see [50] and references therein). However, it should be noted that whereas in these contexts one classifies the symmetries up to the Adjoint action of the full symmetry group, here one can only use the action of the stabiliser groups, since the Killing fields are not necessarily static, and hence may not always define globally valid Hamiltonians.

We also showed that bi-invariant cones in symmetry groups encode the possibility of defining positive energy functionals. Lemma 6 shows their necessity in classical theories, while Theorem 4 shows their necessity and sufficiency for quantum theories with

certain symmetry groups. As we said, more general results relating bi-invariant cones to unitary highest weight representations can be found in [19, Part D]. This approach yields a geometrical description of the set positive energy operators as corresponding to an Adjoint invariant cone in the symmetry group. The fact that symmetries play a major rôle in the problem of positive energy is not new, but our achievement here is relate it to infinitesimal causal structures: Theorem 5 shows that for non-compact simple symmetry groups, spacetime staticity implies the existence of positive energy representations. This should have applications in understanding better the physics of unitary highest weight representations, which commonly arise, for example, in supergravity theories [48]. One should not forget that positive observable-functionals remain positive under changes of observers, and such observables seem less necessary whenever there is a notion of localised states in the theory. The existence of observer-independent future-directions relates to that of positive energy functionals simply because observer-independence is encoded as Adjoint invariance in the symmetry group.

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